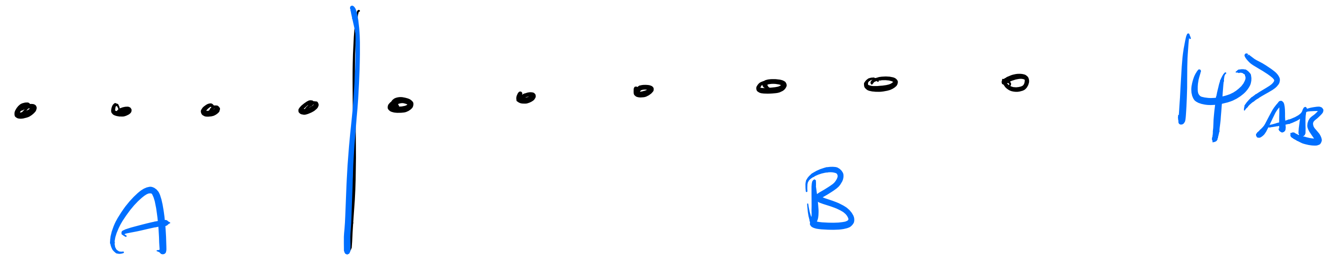
2. Truncation of the bond dimension & approximability by MPS

Can we improve this for 1D states which obey an area law?

Area law in 1D for bipartition:



$$E(A:B) = S(\mathrm{tr}_B |\psi\rangle\langle\psi|) \leq \text{const.} \quad \forall \text{ cuts}.$$

a) 1st step: Area law for 0-Rényi entropy

$$S_0(\rho) = \log \operatorname{rank} \rho,$$

i.e. $\operatorname{rank} \mathrm{tr}_B |\psi\rangle\langle\psi| \leq \chi.$

$\Longleftrightarrow$ Schmidt decomposition at each cut k,

$$|\psi\rangle = \sum_{i=1}^{D_k} s_i |l_i\rangle |r_i\rangle$$

has only χ non-zero terms, $D_k \leq \chi \;\forall k$!

$\Longrightarrow$ In each step of the previous construction, the dimensions D_k of the matrices are $D_k \leq \chi$ (or, alternatively: The rank of the SVD is $\leq \chi$.)

$\Longrightarrow$ For 1D states $|\psi\rangle$ with an area law for the 0-Rényi entropy, $S_0(\rho_A) \leq \log \chi$,

the exact MPS decomposition of $|\psi\rangle$ has bond dimension $\leq \chi$, i.e.,

$$|\psi\rangle = \sum_{i_1 \ldots i_N} U^{(1)}_{i_1} U^{(2)}_{i_2} \cdot \ldots \cdot U^{(N)}_{i_N} |i_1, \ldots, i_N\rangle,$$

with $\mathcal{U}^{(k)}_{i_k,}$ $2 \le k \le N-1$ $\chi \times \chi$ - matrices.

(Note: If a D_k is smaller than χ, we can always pad it with zeros to obtain $\chi \times \chi$ matrices everywhere - if we want.

More generally, "bond dimension χ" should generally be read as "bond dimension at most χ".)

b) What if we have an area law for an α - Rényi entropy?

Consider $\alpha < 1$ ("entropy cares more about # than value of probabilities").

Exercise Sheet 1, Problem 2:

(see also https://arxiv.org/abs/cond-mat/0505140)

Theorem:

Let

not really, but any # is ok.

$$|\psi\rangle = \sum_{i=1}^{\infty} s_i |l_i\rangle_A |r_i\rangle_B ,$$

Schmidt dec.

$$|\psi(\chi)\rangle := \sum_{i=1}^{\chi} s_i |l_i\rangle_A |r_i\rangle_B ,$$

$$|\hat{\psi}(\chi)\rangle := \frac{|\psi(\chi)\rangle}{\| |\psi(\chi)\rangle \|} ,$$

and $\varepsilon(\chi) = \| |\psi\rangle - |\hat{\psi}(\chi)\rangle \|^2 .$

Then, for any $0 < \alpha < 1$,

$$\varepsilon(\chi) \leq \frac{1}{\chi^{\eta_\alpha}} c_\alpha e^{\eta_\alpha S_\alpha(\mathrm{tr}_B |\psi\rangle\langle\psi|)}$$

with $c_\alpha = \alpha(1-\alpha)^{\eta_\alpha}$; $\eta_\alpha = \frac{1-\alpha}{\alpha}$

That is, if the (α-Renyi) entanglement entropy of a bipartite state is bounded, we can approximate it faithfully with a state with small Schmidt rank χ — here, "faithfully" means "with an error which vanishes polynomially in $1/\chi$ (as opposed to exponentially)".

Differently speaking, to get an accuracy $\frac{1}{\varepsilon}$, we only need to take $\mathrm{poly}\left(\frac{1}{\varepsilon}\right)$ parameters (as opposed to $\exp\left(\frac{1}{\varepsilon}\right)$).

We can now use this approximation in every step of the iterative decomposition — it can be shown (exercise sheet 2) that the error is sub-additive, i.e., if in each step,

$$\| |\psi\rangle - |\hat{\psi}(x)\rangle \|^2 \leq \varepsilon$$

at cut k

$$\Longrightarrow \quad \| |\psi\rangle - |\hat{\phi}(x)\rangle \|^2 \leq N \cdot \varepsilon$$

State where we have used the truncated Schmidt dec. at all cuts.

<u>Theorem</u>: Let $|\psi\rangle \in (\mathbb{C}^d)^{\otimes N}$ be a state of a 1D spin chain which obeys an area law for the α-Renyi entropy, $\alpha < 1$, i.e.

$$\forall L: \; S_\alpha(\rho_{1\ldots L}) = S_\alpha(\mathrm{tr}_{L+1,\ldots,N} |\psi\rangle\langle\psi|) \leq S_{max}$$

Then, $|\psi\rangle$ is well approximated by an MPS $|\phi_D\rangle$ with bond dimension D in the following sense: In order to get an approximation with error

$$\varepsilon := \| |\psi\rangle - |\phi_D\rangle \|^2,$$

the required bond dimension is

$$D = c'_\alpha \frac{1}{(\varepsilon/2N)^{1/1-\alpha}} e^{S_{max}},$$

with $c'_\alpha = \alpha^{\alpha/1-\alpha}(1-\alpha)$, that is, the required D – and thus the number of parameters needed – scales <u>polynomially</u> with the <u>desired accuracy</u> and the <u>system size</u>.

Apply this to approximate ground states of local gapped Hamiltonians.

What is the scaling with the gap?

<u>Hastings' proof</u> (2007)

https://arxiv.org/abs/0705.2024

$$S_{max} \propto e^{c \cdot v/\Delta}$$

(where v is the so-called Lieb-Robinson velocity)

<u>Arad, Kitaev, Landau, Vazirani</u> (2013)

https://arxiv.org/abs/1301.1162

$$S_{max} \propto \frac{1}{\Delta}$$

(<u>Note</u>: Both results also directly bounds the weight in the tail of the Schmidt coefficients, and thus directly yield an approximability result.)