5. Examples

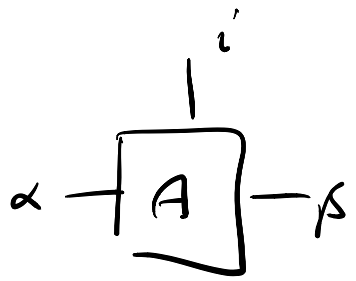
a) The GHZ state

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|00\ldots0\rangle + |11\ldots1\rangle\right) \qquad (d=2)$$

PBC transl. inv. MPS:

$$A^0 = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = |0\rangle\langle 0|$$

$$A^1 = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = |1\rangle\langle 1|$$

or 

$= \delta_{i\alpha\beta}$

we have $A^0 A^0 = A^0$

$$A^1 A^1 = A^1$$

$$A^0 A^1 = 0$$

$$\Rightarrow \quad \mathrm{tr}\left[A^{i_1} A^{i_2} \ldots A^{i_N}\right] = \begin{cases} 1 & i_1 = i_2 = \ldots = i_N \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow |\psi\rangle = \sum \mathrm{tr}\left[A^{i_1} A^{i_2} \cdots A^{i_N}\right] |i_1, \ldots, i_N\rangle$$

$$= |0 \ldots 0\rangle + |1 \ldots 1\rangle$$

GHZ state up to normalization.

Can normalize e.g. by setting $A^{i_1,(1)} = \frac{1}{\sqrt{2}} A^{i_1}$, $A^{i_k,(k)} = A^{i_k}$. But: this breaks tr. inv. of representation.

Note: MPS are generally not normalized (cf. later).

Can also be written with OBC:

$$\langle +| A^{i_1} \cdots\cdots A^{i_N} |+\rangle =$$

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$= \delta_{i_1 \ldots i_N} \langle +| A^{i_1} |+\rangle = \frac{1}{2}.$$

$\Longrightarrow$ MPS with

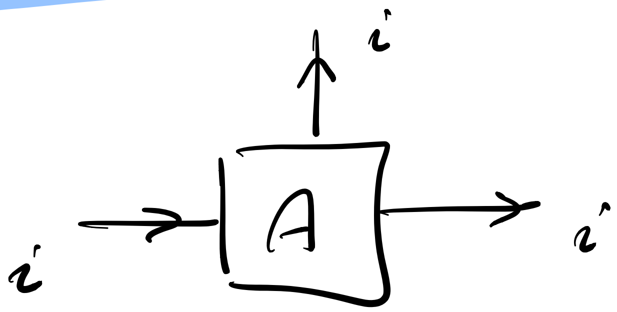
$$B^{i_1,(1)} := \sqrt{2}\langle +| A^{i_1}$$

$$B^{i_k,(k)} := A^k$$

$$B^{i_N,(N)} := A^{i_N} |+\rangle$$

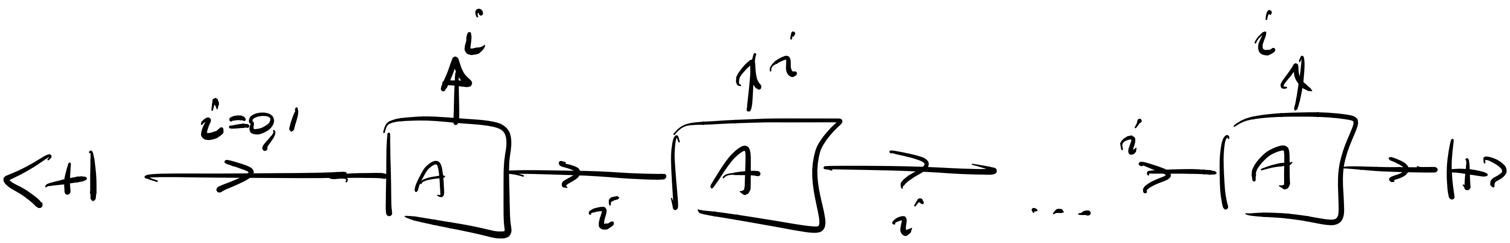
gives OBC rep. of GHZ state.

"Agent" interpretation:



A takes input i, and outputs i as a physical system, and i as a virtual system,

Total GHZ state in OBC rep.:



Each process has an amplitude associated to it; the total amplitude is the product of the amplitudes (cf. path integral).

Sometimes this gives a very natural perspective on MPS.

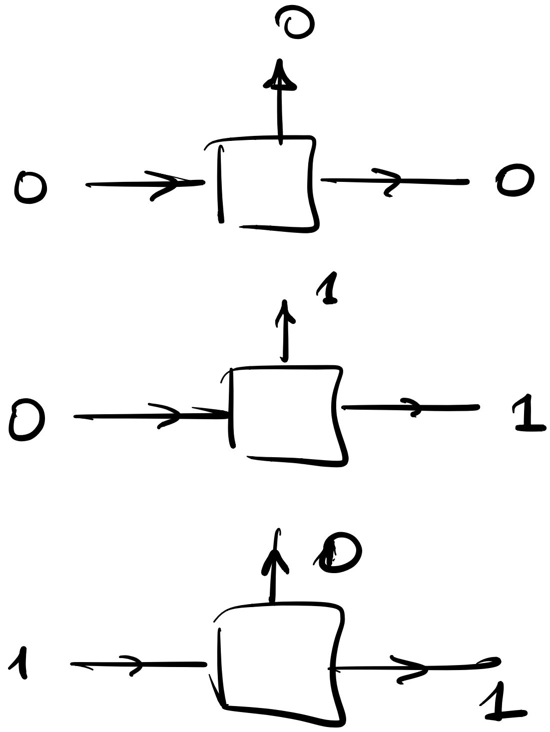
5) The W state

$$|W\rangle = \frac{1}{\sqrt{N}}\left(|100\ldots0\rangle + |010\ldots0\rangle + \ldots + |00\ldots01\rangle\right)$$

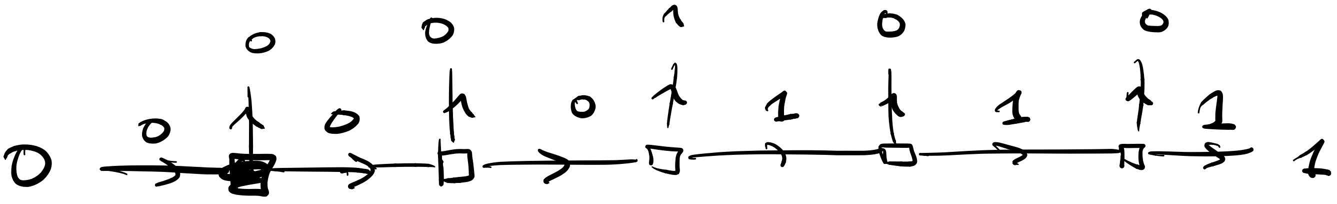
Agent picture:

1. Start with 0.

2. valid transitions:



③ final configuration: 1



$$A^0 = |0\rangle\langle 0| + |1\rangle\langle 1| = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \mathbb{1}$$

$$A^1 = |0\rangle\langle 1| = \sigma^-$$

$$|\psi\rangle = \sum_{i_1,\dots,i_N} \underbrace{\langle 0|A^{i_1}}_{\equiv B^{i_1,(1)}} \underbrace{A^{i_2}\cdots\cdots}_{\equiv B^{i_k,(k)}} \underbrace{A^{i_N}|1\rangle}_{\equiv B^{i_N,(N)}} |i_1,\dots,i_N\rangle$$

$(\sigma^-)^2 = 0$, $\langle 0|\sigma^-|1\rangle = 1$.

$\Rightarrow |\psi\rangle \propto |W\rangle$

<u>Note:</u> No tr. inv. PBC rep. of $|W\rangle$ exists, unless D grows with N.

(Any tr. inv. OBC MPS can be transformed in a tr. inv. PBC MPS with $D_{PBC} = N D_{OBC}$.)

c) The cluster state

The cluster state is obtained by acting with

$$CZ_{i,i+1} = \begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&-1\end{pmatrix} \quad \text{on} \quad |+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}},$$

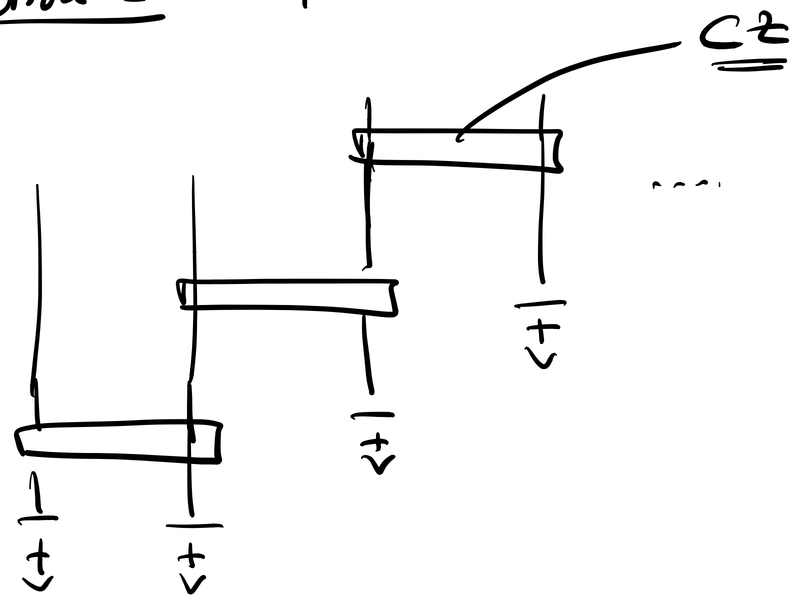
$$|\mathcal{C}\rangle = \prod_{i=1}^{N-1} CZ_{i,i+1} \, |+\rangle^{\otimes N} \quad \text{(w/ OBC).}$$

(Note: All $CZ_{i,i+1}$ commute – any order ok.)

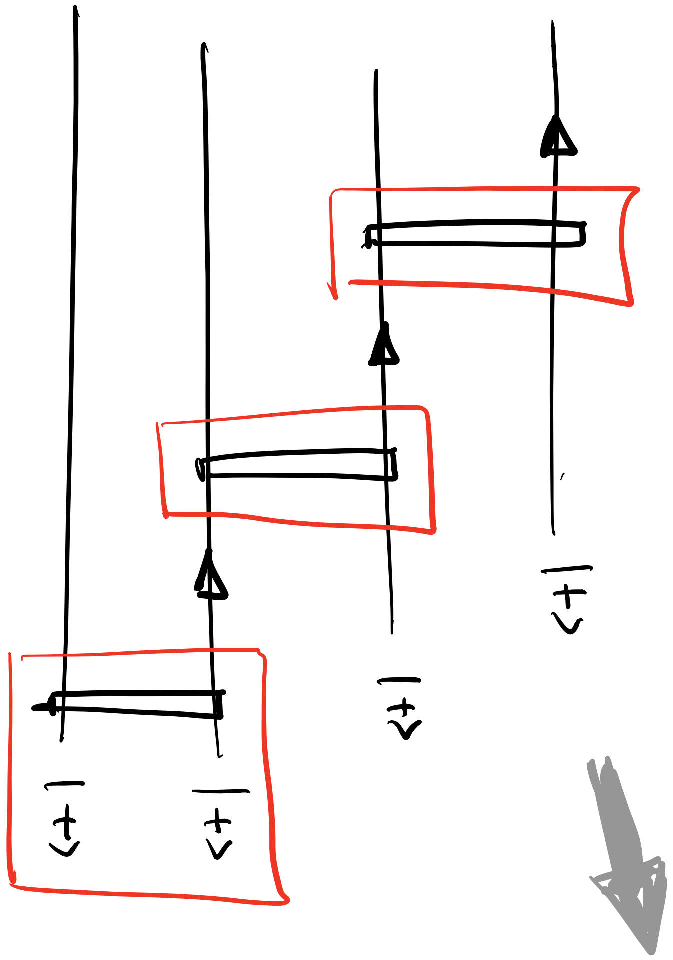
How can we find MPS representation?

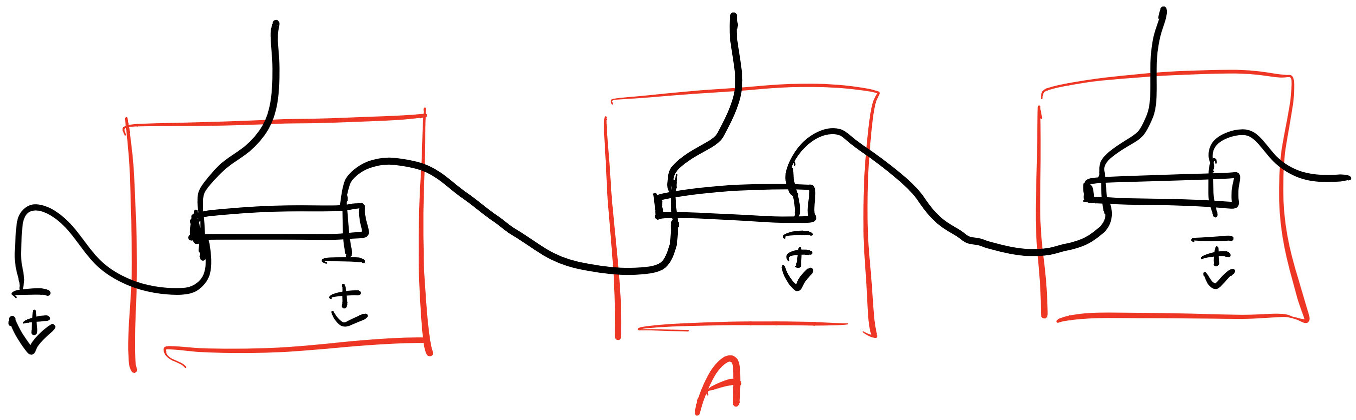
Different approaches possible...

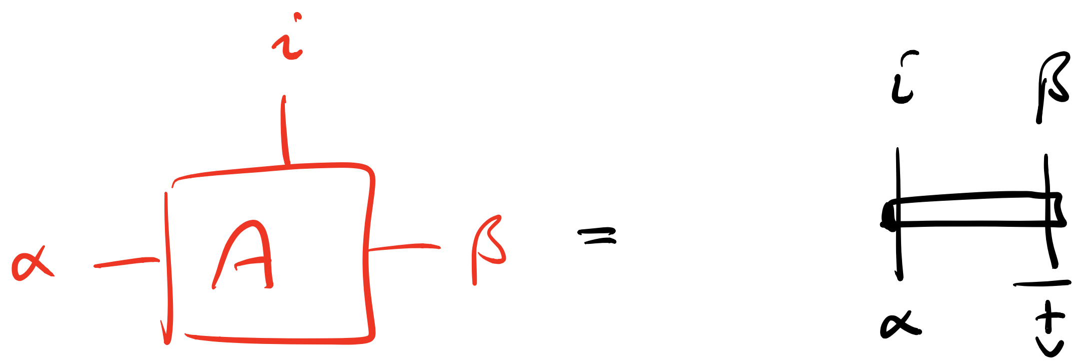
Option 1: $|\mathcal{C}\rangle =$



Adjoint picture:







$$= \langle i| \otimes \langle \beta| \; CZ \; \underbrace{|\alpha\rangle \otimes |+\rangle}_{\frac{|\alpha 0\rangle + |\alpha 1\rangle}{\sqrt{2}}}$$

$\alpha = 0$

$$\frac{|\alpha 0\rangle + |\alpha 1\rangle}{\sqrt{2}}$$

$\langle i, \beta | \dots \rangle \Rightarrow$

$$\frac{\delta_{\alpha i}}{\sqrt{2}} \cdot (+1)^{\beta}$$

for $\alpha = 0$

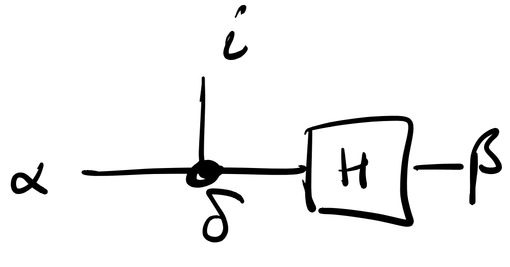
$\alpha = 1$

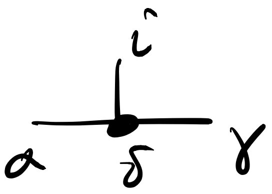
$$\frac{|\alpha 0\rangle - |\alpha 1\rangle}{\sqrt{2}}$$

$\langle i, \beta | \dots \rangle \Rightarrow$

$$\frac{\delta_{\alpha i}}{\sqrt{2}} (-1)^{\beta}$$

for $\alpha = 1$



with 

the δ-tensor:

$$\delta_{\alpha i \gamma} = \begin{cases} 1 & \alpha = i = \gamma \\ 0 & \text{otherwise} \end{cases}$$

and $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = |0\rangle\langle +| + |1\rangle\langle -|$

the Hadamard matrix/transformation.

i.e.:

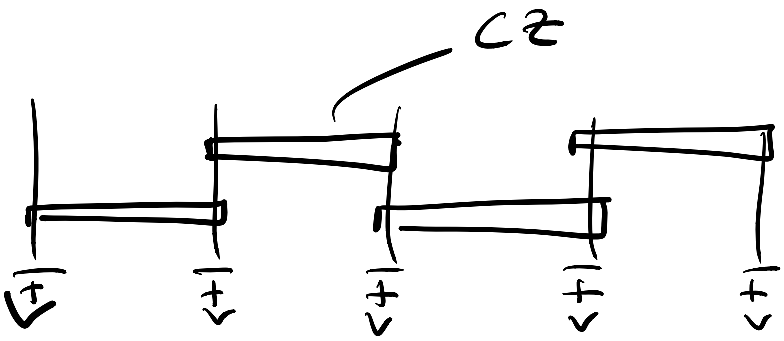
or

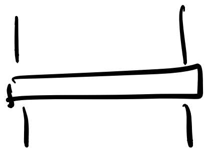
$$A^0 = |0\rangle\langle 0| \cdot H = |0\rangle\langle +|$$

$$A^1 = |1\rangle\langle 1|\, H = |1\rangle\langle -|.$$

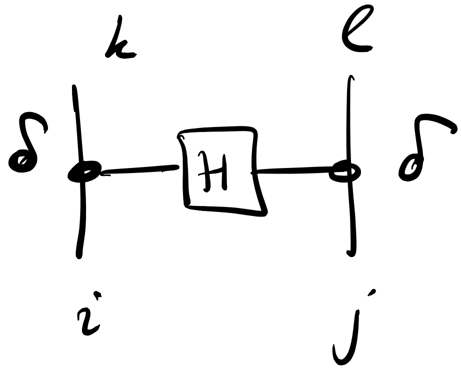
$$|\ell\ell\rangle = \sum_{i_1,\dots,i_N} \langle +| A^{i_1} \dots A^{i_N} |0\rangle\, |i_1,\dots,i_N\rangle$$

Option 2:

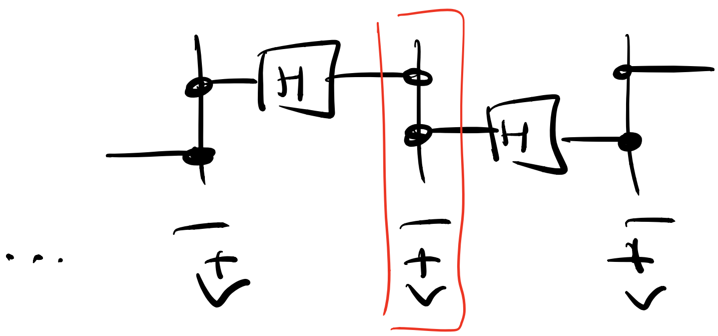
$|\psi\rangle =$ 

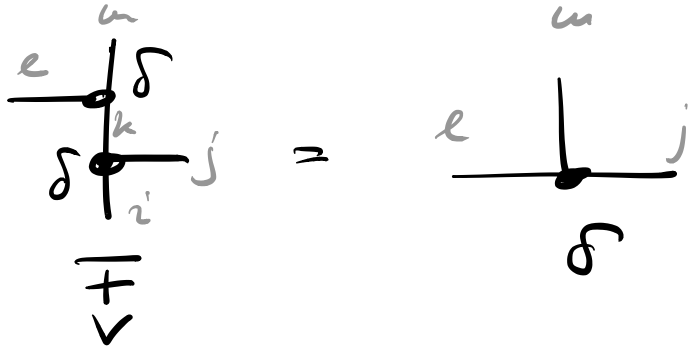


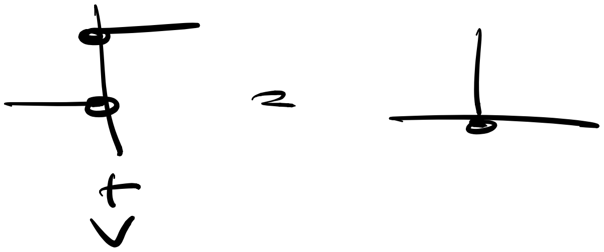
$$= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

$=$ 

$$= \delta_{ik}\,\delta_{jl}\,\underbrace{(H)_{ij}}_{= (-1)^{i\cdot j}}$$ ✓

$\Rightarrow |\psi\rangle =$ 

and  !

(and same for )

$\Longrightarrow$ same MPS representation

(but this also works for PBC).