IV. Solvable models and the classification of phases

1. The AKLT model

a) Construction

Let $|\omega\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right) \in \mathbb{C}^2 \otimes \mathbb{C}^2$

$|\omega\rangle$ is $SU(2)$-invariant:

$$(u \otimes u)|\omega\rangle = |\omega\rangle \qquad \forall\, u \in SU(2)$$

The space $\mathbb{C}^2 \otimes \mathbb{C}^2$ can be decomposed naturally into a anti-sym. and sym. space,

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathcal{A} \oplus \mathcal{S},$$

with

$$\mathcal{S} = \text{span}\{\, |S=1, m=+1\rangle,\ |S=1, m=0\rangle,\ |S=1, m=1\rangle \,\},$$

where

$$|S=1; m=+1\rangle = |00\rangle$$

$$|S=1, m=0\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$$

$$|S=1, m=-1\rangle = |11\rangle$$

and $A = \text{span}\{|s=0, m=0\rangle\}$

$$|s=0, m=0\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

Thus naturally decomposes

$$u \otimes u \cong 1 \oplus V_u$$

spin-0 space

spin-1 space & action

Now define

$$P : \mathbb{C}^2 \otimes \mathbb{C}^2 \longrightarrow \mathbb{C}^3$$

$$P = |+1\rangle\langle s=1, m=+1| + |0\rangle\langle s=1, m=0| + |-1\rangle\langle s=1, m=-1|$$

the isometry projecting onto the spin-1 space.

In particular,

$$P(u \otimes u) = V_u P.$$

(Note: Up to a phase ± 1, $u \in SU(2)$ can be understood as the rotation of a spin-$\frac{1}{2}$ particle by an angle of $|\vec{\theta}|$ about an axis $\vec{n} = \vec{\theta}/|\vec{\theta}|$ (i.e., $\vec{\theta} \in SO(3)$):

$$u = u(\vec{\theta}) = e^{i\vec{\theta}\cdot\vec{S}^{1/2}},$$

with $\vec{S}^{1/2} = (\frac{1}{2}\sigma_x, \frac{1}{2}\sigma_y, \frac{1}{2}\sigma_z)$ the spin-1/2 operators.

Then, similarly, $V_{\vec{\theta}} = V_{u(\vec{\theta})} = e^{i\vec{\theta}\cdot\vec{S}^1}$,

with $\vec{S}^1 = (S_x^1, S_y^1, S_z^1)$ the spin-1 operators.)

Now consider a chain of $2N$ spins $\mathbb{C}^2$, and construct the state

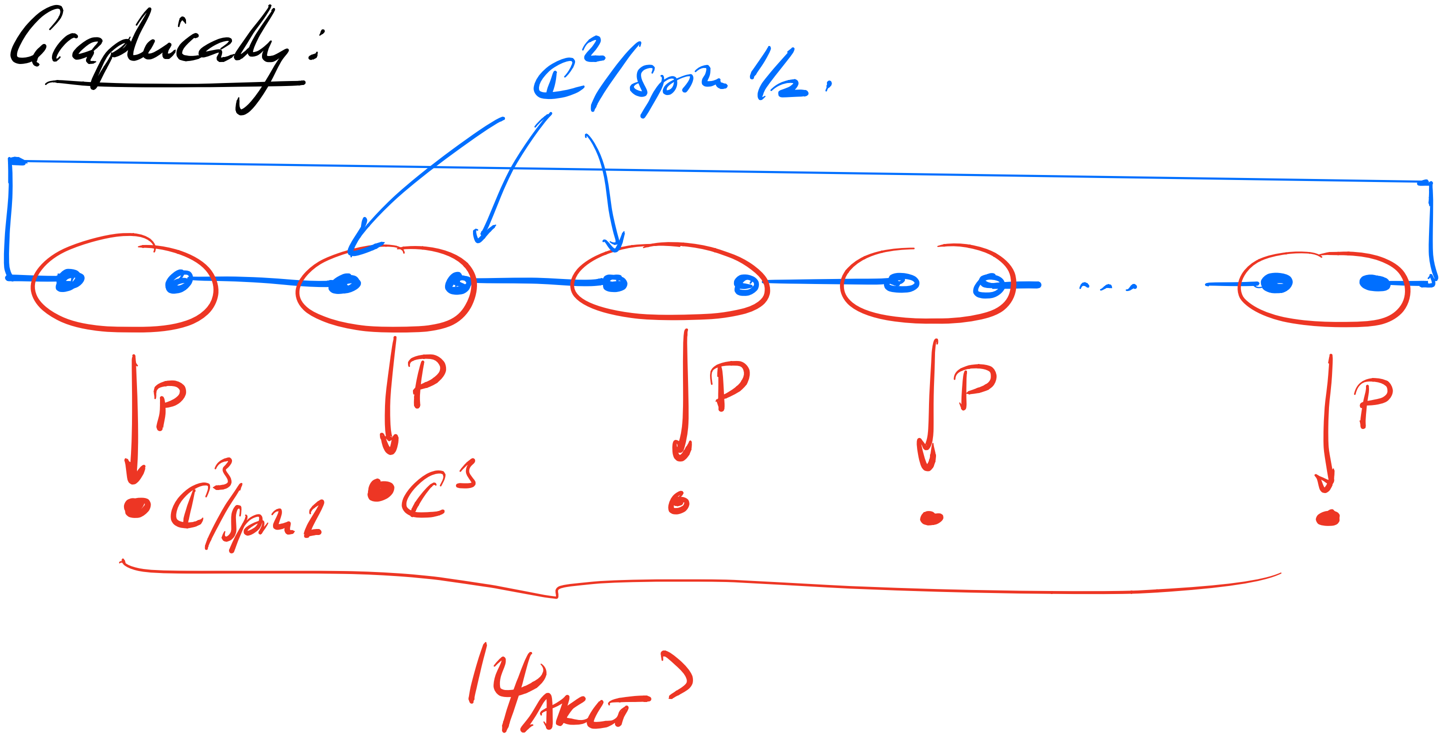
$$|\Psi_{AKLT}\rangle = \left(P_{12} \otimes P_{34} \otimes \ldots \otimes P_{N-1,N}\right)\left(|\omega\rangle_{23} \otimes |\omega\rangle_{45} \otimes \ldots \otimes |\omega\rangle_{N-2,N-1} \otimes |\omega\rangle_{N,1}\right)$$

spins on which it acts

$$= \text{"}\left(P^{\otimes N}\right)\left(|\omega\rangle^{\otimes N}\right)\text{"}.$$

This is the **AKLT state**.
(Affleck, Kennedy, Lieb, Tasaki)

Graphically:



The AKLT state is rotationally (i.e. SO(3)) invariant:

$$V_{\vec{\theta}}^{\otimes N}\,|\psi_{AKLT}\rangle = V_{\vec{\theta}}^{\otimes N}\left(P^{\otimes N}\right)\left(|\omega\rangle^{\otimes N}\right)$$

$$= \left(P\left(u_{\vec{\theta}}\otimes u_{\vec{\theta}}\right)\right)^{\otimes N}\,|\omega\rangle^{\otimes N}$$

$$= P^{\otimes N} \, u_{\vec{\theta}}^{\otimes 2N} \, |\omega\rangle^{\otimes N}$$

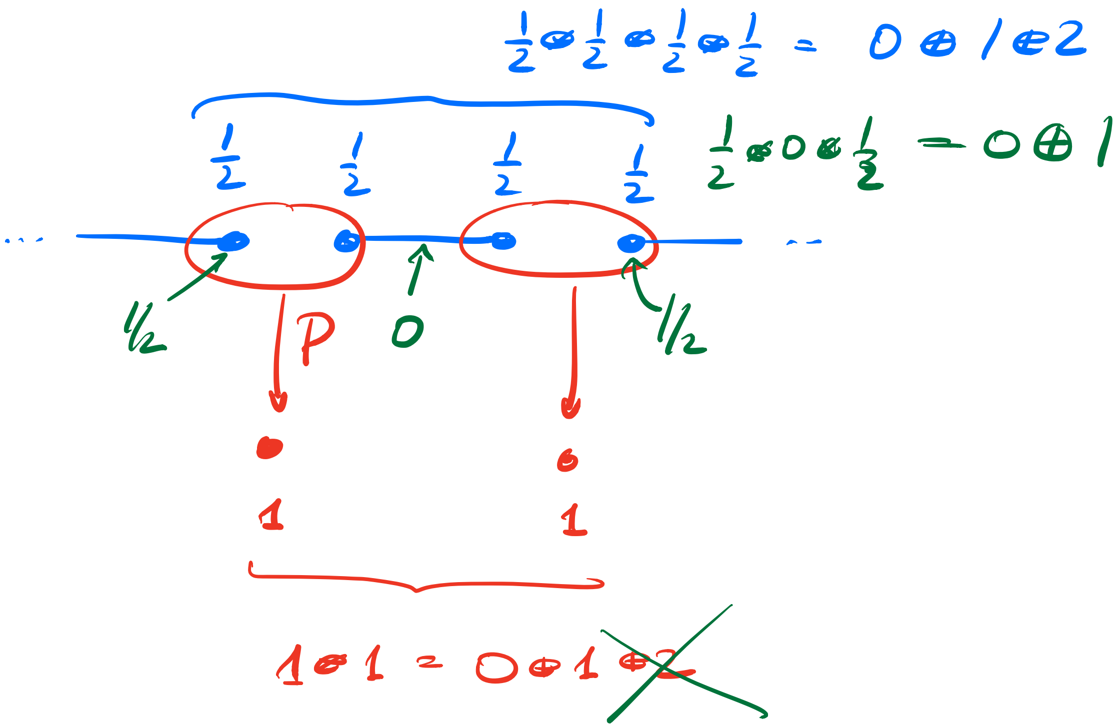
$$= P^{\otimes N} \Big(\underbrace{(u_{\vec{\theta}} \otimes u_{\vec{\theta}}) \, |\omega\rangle}_{= |\omega\rangle} \Big)^{\otimes N}$$

$$= P^{\otimes N} \, |\omega\rangle^{\otimes N} = \underline{|\psi_{AKLT}\rangle}$$

b) Parent Hamiltonians

Does the AKLT model appear as ground state of some local Hamiltonian?

Consider 2 consecutive sites on the AKLT chain.



Auxiliary spin $\frac{1}{2}$ - deg. of freedom are in spin $\frac{1}{2}, 0, \frac{1}{2}$, respectively,

and P does not change spin:

$\Rightarrow$ final state on 2 sites (i.e., reduced density matrix after proj. out the rest of the chain) cannot have spin 2!

i.e.: $\mathrm{tr}_{3,4,\dots,N}(|\psi_{AKLT}\rangle\langle\psi_{AKLT}|) = \rho_{12}$

is supported on space with spin $= 0, 1$.

(i.e.: $\mathbb{C}^3 \otimes \mathbb{C}^3 = \mathcal{H}_{S=0} \oplus \mathcal{H}_{S=1} \oplus \mathcal{H}_{S=2}$)

and $\mathrm{supp}(\rho_{12}) \subset \mathcal{H}_{S=0} \oplus \mathcal{H}_{S=1}$.)

Define Hamiltonian acting on sites 1, 2:

$$h_{12} = \Pi_{S=2},$$

where $\Pi_{S=2}$ is the projector onto the spin-2 space (i.e. $\Pi_{S=2}$ projects onto $\mathcal{H}_{S=2}$).

Then, $\langle \psi_{AKLT} | h_{12} | \psi_{AKLT} \rangle$

$$= tr [\rho_{12} h_{12}] = 0.$$

Since $eig(h_{12}) = \{0, 1\}$, this means $|\psi_{AKLT}\rangle$ is an eigenstate of h_{12} with eigenvalue 0:

$$h_{12} |\psi_{AKLT}\rangle = 0.$$

We can use the same argument for any 2 adjacent sites $i, i+1$:

$$h_{i,i+1} = (\Pi_{S=2})_{i,i+1}$$

proj. on $S=2$ subspace at sites $i, i+1$.

$$h_{i,i+1} |\psi_{AKLT}\rangle = 0.$$

Then, for $H_{AKLT} = \sum_{i=1}^{N} h_{i;i+1}$, we have:

$$H_{AKLT} = \sum_{i=1}^{N} \underbrace{h_{i;i+1}}_{\geq 0} \geq 0$$

(i.e. H only has eigenvalues ≥ 0), and

$$H_{AKLT} |\psi_{AKLT}\rangle = \sum_{i=1}^{N} \underbrace{h_{i;i+1} |\psi_{AKLT}\rangle}_{=0} = 0$$

$\Longrightarrow$ $|\psi_{AKLT}\rangle$ is a ground state of H_{AKLT}!

Note: A Hamiltonian $H = \sum h_{i;i+1}$ where the ground state minimizes the energy of each $h_{i;i+1}$ individually is called frustration free.

Moreover, H_{AKLT} is rotationally invariant:

$$\left(V_{\vec{\theta}} \otimes V_{\vec{\theta}}\right) h_{12} = h_{12} \left(V_{\vec{\theta}} \otimes V_{\vec{\theta}}\right),$$

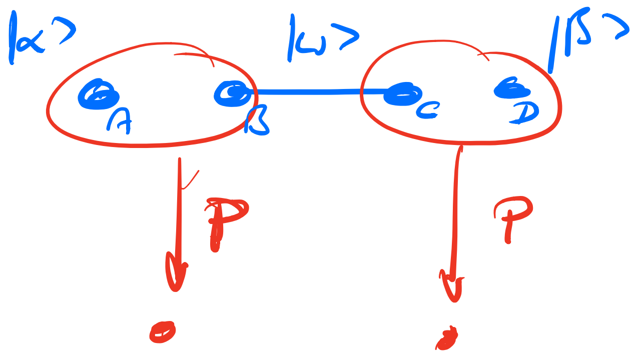
since a subspace of constant spin is invariant under rotations (but not the vectors in the space!).

Note:

$$H_{AKLT} = \sum_{i=1}^{N} \left(\frac{1}{2}(\vec{S}_i \cdot \vec{S}_{i+1}) + \frac{1}{6}(\vec{S}_i \cdot \vec{S}_{i+1})^2 + \frac{1}{3} \right).$$

Theorem: $|\psi_{AKLT}\rangle$ is the unique ground state of H_{AKLT}.

Proof: First, consider 2 sites of the AKLT chain:



Define the total map

$$Q := P_{AB} \otimes P_{CD} \quad |\omega\rangle_{BC}$$

which maps $|\alpha\rangle_A$ & $|\beta\rangle_D$ to the two spin-1s.

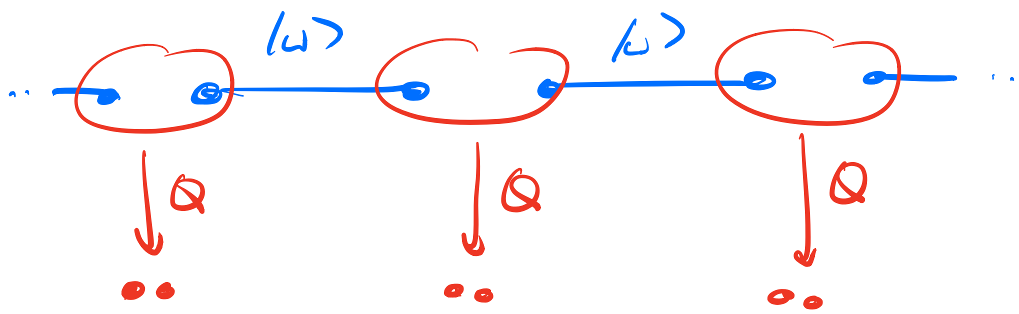
Lemma: Q is injective.

Proof: Homework.

This implies that Q has a left-inverse Q^{-1}:

$$Q^{-1}Q = \mathbb{1}.$$

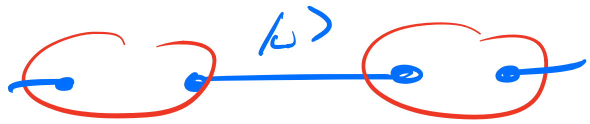
The AKLT chain on N sites can be replaced by



on $N/2$ sites, i.e.

$$|\psi_{AKLT}\rangle = Q^{\otimes N/2}|\Omega\rangle, \quad |\Omega\rangle = |\omega\rangle^{\otimes N/2}.$$

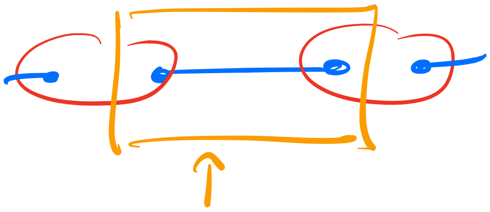
$|\Omega\rangle$ is trivially the unique ground state of $H_\Omega = \sum k_{i,i+1}$,



$$k_{i,i+1} = \mathbb{1} \otimes (\mathbb{1} - |\omega\rangle\langle\omega|) \otimes \mathbb{1}$$

$$H_\Omega = \sum k_{i,i+1} \geq 0 .$$

Clearly: $H_\Omega |\phi\rangle = 0 \iff |\phi\rangle = |\Omega\rangle$

(As $k_{i,i+1} |\phi\rangle = 0 \Rightarrow$ on the two central sites , $|\phi\rangle$ equals $|\omega\rangle$

$\Rightarrow k_{i,i+1} |\phi\rangle = 0 \ \forall i$: $|\phi\rangle = |\omega\rangle^{\otimes N}$).

Now consider

$$\tilde{k}_{i,i+1} = \underbrace{\left(Q_i^{-1} \otimes Q_{i+1}^{-1}\right)^\dagger \left(k_{i,i+1}\right) \left(Q_i^{-1} \otimes Q_{i+1}^{-1}\right)}_{\geq 0}$$

$$+ \underbrace{\left(\mathbb{1} - \Pi_{\mathrm{Im}(Q)}\right) \otimes \mathbb{1}}_{\geq 0}$$

$$+ \underbrace{\mathbb{1} \otimes \left(\mathbb{1} - \Pi_{\mathrm{Im}(Q)}\right)}_{\geq 0} .$$

and $\tilde{H} = \sum \tilde{k}_{i,i+1} \geq 0 .$

Then:

$$\tilde{k}_{i,i+1} |\psi_{AKLT}\rangle = \tilde{k}_{i,i+1} (Q^{\otimes N/2}) |\Omega\rangle$$

$$= (Q_i^{-1} \otimes Q_{i+1}^{-1})^\dagger (k_{i,i+1}) (Q_i^{-1} \otimes Q_{i+1}^{-1}) (Q_1 \otimes \ldots \otimes Q_i \otimes Q_{i+1} \otimes \ldots) |\Omega\rangle$$

$$+ \underbrace{((\mathbb{1} - \Pi_{\ker(Q)}) \otimes \mathbb{1}) (Q^{\otimes N/2}}_{=0} |\Omega\rangle) + \underbrace{\ldots}_{=0}$$

$$= (Q_1 \otimes \ldots \otimes Q_{i-1} \otimes Q_i^{-1\dagger} \otimes Q_{i+1}^{-1\dagger} \otimes Q_{i+2} \otimes \ldots \otimes Q_{N/2}) \underbrace{(k_{i,i+1}) |\Omega\rangle}_{=0}$$

$$= 0$$

$$\Rightarrow \tilde{H} |\psi_{AKLT}\rangle = 0$$

$\Rightarrow |\psi_{AKLT}\rangle$ ground state of $\tilde{H}$.

Now let $|\tilde{\phi}\rangle$ s.th. $\tilde{H} |\tilde{\phi}\rangle = 0$:

$$\Rightarrow \forall i \quad \tilde{k}_{i,i+1} |\tilde{\phi}\rangle = 0$$

$\Rightarrow \forall i : \mathbb{1} \otimes (\mathbb{1} - \Pi_{\ker(Q)}) \otimes \mathbb{1} \; |\tilde{\phi}\rangle = 0$, i.e.

$$|\tilde{\phi}\rangle \in (\mathrm{Im}(Q))^{\otimes N/2},$$

and $$(Q_i^{-1} \otimes Q_{i+1}^{-1})^{+} (k_{i,i+1}) (Q_i^{-1} \otimes Q_{i+1}^{-1}) |\tilde{\phi}\rangle = 0$$

$$\Rightarrow \forall i: \; k_{i,i+1} (Q_i^{-1} \otimes Q_{i+1}^{-1}) |\tilde{\phi}\rangle = 0$$

$$\Rightarrow \forall i: \; k_{i,i+1} (Q^{-1})^{\otimes N/2} |\tilde{\phi}\rangle = 0$$

$\xRightarrow[\text{g.s. of } \sum k_{i,i+1}]{\text{uniqueness of}}$ $$(Q^{-1})^{\otimes N/2} |\tilde{\phi}\rangle = |\Omega\rangle$$

and since $QQ^{-1} = \mathbb{1}_{\mathrm{Im}(Q)}$:

$$\Longrightarrow \quad |\tilde{\phi}\rangle = Q^{\otimes N/2} |\Omega\rangle$$

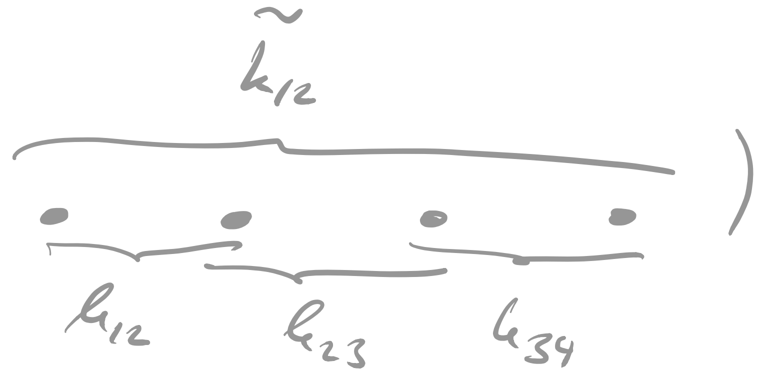
$\Rightarrow$ $|\psi_{AKLT}\rangle$ is unique ground state of $\tilde{H} = \sum \tilde{k}_{i,i+1}$.

Last missing step:

AKLT Ham.

Lemma: $\ker(\tilde{h}_{12}) = \ker(h_{12} + h_{23} + h_{34})$

(Note: Due to blocking:



Proof: E.g. brute force on a computer or by hand ($\rightarrow$ HW).

Due to the frustration-free property – a state is a ground state if & only if it is in $\ker(\tilde{h})$ or $\ker(h)$ for all terms – this implies that $|\psi_{AKLT}\rangle$ is the unique ground state of H_{AKLT} as well.

$\Rightarrow$ This concludes the proof!

Theorem H_{AKLT} has a gap.

Proof:

We know: $H_{AKLT}|\Psi_{AKLT}\rangle = 0$ is (unique) ground state.

Lemma: If H has a ground state energy 0, then

$$H^2 \geq \gamma H \iff H \text{ has a gap (no eigenvalues) between } 0 \text{ and } \gamma.$$

Proof:

Write $H = \sum E_n |n\rangle\langle n|$ (eigenval. dec.).

Then,

$$H^2 = \sum E_n^2 |n\rangle\langle n| \geq \gamma E_n |n\rangle\langle n|$$

if and only if $E_n^2 \geq \gamma E_n \;\; \forall n$,

i.e., $E_n \notin (0,\gamma) \;\; \forall n$ ☒

Lemma: (Knabe bound)

Let $H = \sum_{i=1}^{N} h_{i,i+1}$ be a tr. inv., periodic, frustration free Ham. w/ $h_{i,i+1}^2 = h_{i,i+1}$,

and $H'_n = \sum_{i=1}^{n} h_{i,i+1}$ with open bnd.

$(n \geq 2)$.

Let $\Delta(H)$ and $\Delta(H'_n)$ denote the gaps of H & H'_n, respectively.

Then,

$$\Delta(H) \geq \frac{n}{n-1}\left(\Delta(H'_n) - \frac{1}{n}\right).$$

In particular, if $\Delta(H'_n) > \frac{1}{n}$ for some $n \geq 2$, then H is gapped.

Proof: We have

$$(H_n')^2 \geq \Delta(H_n') H_n'$$

$$\Rightarrow \sum_{i,j=1}^{n} h_{i,i+1} h_{j,j+1} \geq \Delta(H_n') \sum_{i=1}^{n} h_{i,i+1} \quad (*)$$

— This holds of course for every $\sum_{i,j=k}^{k+n-1}$ —

Summing all k:

$\Longrightarrow$

$$\sum_{|i-j|<n} (n - |i-j|) h_{i,i+1} h_{j,j+1} \geq \Delta(H_n') \cdot n \cdot \underbrace{\sum_{i=1}^{N} h_{i,i+1}}_{= H}$$

$|i-j| \geq 2$: $h_{i,i+1} h_{j,j+1} \geq 0$

$\Rightarrow$ can add more on LHS

to get weight $n-1$.

$|i-j| = 0$: $h_{i,i+1}^2 = h_{i,i+1}$

$\Longrightarrow$

$$\underbrace{\sum h_{i,i+1}^2}_{=H} + \underbrace{\sum_{i,j} (n-1)\, h_{i,i+1} h_{j,j+1}}_{=(n-1)H^2} \geq \Delta(H_n')\, n\, H$$

$$\Longrightarrow \quad (n-1)\, H^2 \geq \left(n \Delta(H_n') - 1\right) H$$

$$\Longrightarrow \quad H^2 \geq \frac{n\Delta(H_n') - 1}{n-1}\, H$$

$$\Longrightarrow \quad \Delta(H) \geq \frac{n}{n-1}\left(\Delta(H_n') - \frac{1}{n}\right).$$

∎

We can now verify numerically for the AKLT Hamiltonian that

$$\Delta(H_n') > \frac{1}{n} \quad \text{for} \quad n = 3.$$

c) The Haldane conjecture

Consider spin - S Heisenberg model:

$$H = \sum_{i=1}^{N} \vec{S_i} \cdot \vec{S_{i+1}}$$

S = 1/2, 3/2, 5/2, ... :

Lieb-Schultz-Mattis-Theorem ('61):

H is symmetry breaking or gapless — that is, H cannot be gapped with a unique ground state.

S = 0, 1, 2, ... :

Haldane ('83): $H = \sum \vec{S_i} \cdot \vec{S_{i+1}}$ for integer spin has unique ground state + gap.

Argument works via mapping to field theory ("non-linear sigma model") in limit $S \to \infty$ $\Rightarrow$ not fully rigorous.

Argument based on symmetries of model.

AKLT model: Provides a rigorous variant of an integer spin chain with SU(2) symmetry where the gap can be rigorously prove!

d) Fractional edge modes

Consider AKLT Hamiltonian w/ open boundaries:

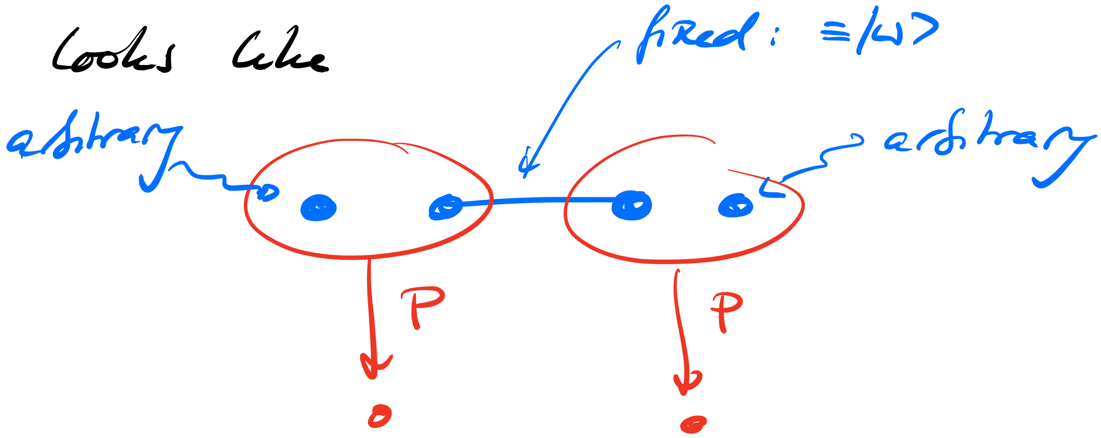
$$H = \sum_{i=1}^{N-1} h_{i,i+1}$$

What are the ground states?

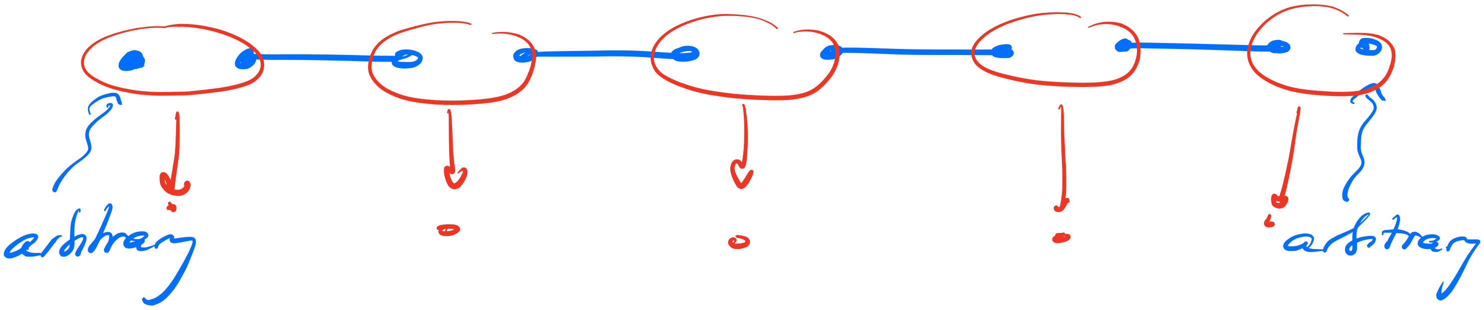
Following earlier proof of uniqueness:

$h_{i,i+1}$ (or its blocked version) make sure that on sites $i, i+1$, the ground state

Looks like



On OBC: Precisely spins on edge remain arbitrary.



→ AKLT chain w/ OBC has 4-fold degen. ground space.

→ Parametrized by a spin-½ degree of freedom ("edge mode") at each boundary.

→ Edge modes localized at boundaries (→ more later) → each edge carries a spin-½ excitation.

This is <u>very surprising</u>: In a spin system, local excitations — created e.g. by S^+ — should have <u>integer spin</u> (as S^+ changes spin by 1).

$\Rightarrow$ The spins "fractionalize" at the edge; and such fractional excitations can only be created <u>in pairs</u>.

Unconventional behavior:

→ points to non-trivial quantum correlations in the system

→ sign of a different type/phase of matter?