Lecture 260070 "Entanglement in quantum many-body systems" – SS 2021

— Exercise Sheet #6 —

Problem 10: AKLT I

Let S^s_α, $s = \frac{1}{2}, 1$, $\alpha = x, y, z$ denote the spin operators for spin s, $|\omega\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, and P the AKLT projector as defined in the lecture.

1. Determine $u_{\vec{\theta}} = \exp(i\vec{\theta} \cdot \vec{S}^{1/2})$. Show that this yields all SU(2) matrices up to a phase ± 1 for $\vec{\theta} \in \mathrm{SO}(3)$ (i.e. $|\vec{\theta}| \leq \pi$, with opposite vectors $\pm\vec{\theta}$ of length $|\vec{\theta}| = \pi$ identified).

2. Show that $(u \otimes u)|\omega\rangle = \omega$ for any $u \in \mathrm{SU}(2)$. (What happens if $u \in \mathrm{U}(2)$?)

3. Show that $P(S^{1/2}_\alpha \otimes \mathbb{1} + \mathbb{1} \otimes S^{1/2}_\alpha) = S^1_\alpha P$.
(If this is not the case, maybe you have used incompatible definitions for S^1 and $S^{1/2}$ – in that case, you should check that the S^1_α obtained here are spin-1 operators as well and update your definition of S^1_α.)
Moreover, show that
$$P^\dagger P(S^{1/2}_\alpha \otimes \mathbb{1} + \mathbb{1} \otimes S^{1/2}_\alpha) = (S^{1/2}_\alpha \otimes \mathbb{1} + \mathbb{1} \otimes S^{1/2}_\alpha)P^\dagger P = S^{1/2}_\alpha \otimes \mathbb{1} + \mathbb{1} \otimes S^{1/2}_\alpha \ .$$

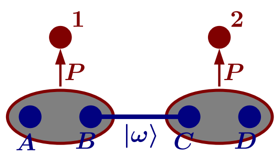
4. Show that $P(u_{\vec{\theta}} \otimes u_{\vec{\theta}}) = \exp(i\vec{\theta} \cdot \vec{S}^1)P$.

5. Check the formula for the AKLT Hamiltonian $h = \Pi_{S=2}$ (the projector onto the spin-2 space on two adjacent sites),
$$h = \tfrac{1}{2}\vec{S}_1 \cdot \vec{S}_2 + \tfrac{1}{6}(\vec{S}_1 \cdot \vec{S}_2)^2 + \tfrac{1}{3} \ .$$
Show that *any* Hamiltonian of the form $w_0\Pi_{S=0} + w_1\Pi_{S=1} + w_2\Pi_{S=2}$ can be expressed in this form with suitable prefactors. (*Hint:* An elegant way is to use that $(\vec{S}_1 + \vec{S}_2)$ is the total spin operator, which fixes the possible values of $(\vec{S}_1 + \vec{S}_2)^2 = \vec{S}_1^2 + \vec{S}_2^2 + 2\vec{S}_1 \cdot \vec{S}_2$ depending on the total spin $S = 0, 1, 2$.)

Problem 11: AKLT II

Let P and $|\omega\rangle$ be as before (i.e., for the AKLT state).

1. Show that the map $Q = (P_{AB} \otimes P_{CD})|\omega\rangle_{BC}$ defined in the lecture,



(1)

which maps the degrees of freedom A and D to the two physical degrees of freedom, is injective. (This can be done analytically or numerically.)

2. Check that the kernel of the Hamiltonian $\tilde{k}_{12}$ on two blocked sites (i.e., four unblocked sites), as constructed in the lecture, equals the kernel of the AKLT Hamiltonian on the same four sites,
$$\ker(\tilde{k}_{12}) = \ker(h_{12} + h_{23} + h_{34}) \ .$$

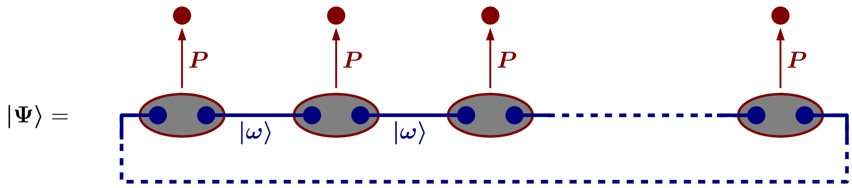
3. Check that the AKLT Hamiltonian on 3 sites (with open boundaries) satisfies the condition of the Knabe bound (i.e., its gap above the zero-energy space is larger than $1/n = 1/3$). (This is best done numerically.) If you want, you can try to push this to more sites, to obtain an as good as possible bound on the gap of the AKLT Hamiltonian.

Problem 12: AKLT construction vs. MPS

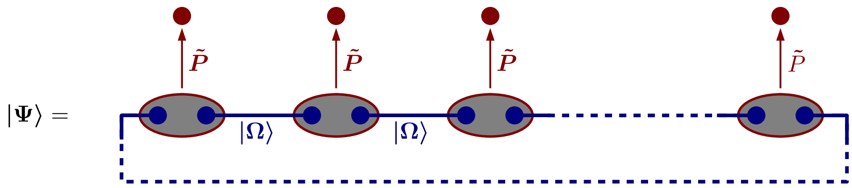
In this problem, we will establish that the AKLT-type construction – starting from a state $|\omega\rangle$ and applying a map P – is in fact equivalent to the MPS construction.

1. Show that *any* state $|\omega\rangle \in \mathbb{C}^D \otimes \mathbb{C}^D$ can be written as $|\omega\rangle = (\mathbb{1} \otimes M)|\Omega\rangle$, with $|\Omega\rangle = \frac{1}{\sqrt{D}} \sum_{i=1}^{D} |i,i\rangle$, with a suitable matrix M.

2. Show that this implies that any state $|\Psi\rangle$ constructed in the same way as the AKLT state,

   where $|\omega\rangle$ and P are *arbitrary*, can be rewritten as

   (where the entangled state now is $|\Omega\rangle$) with a new map $\tilde{P}$. What is $\tilde{P}$?

3. The map $\tilde{P}$ can be written in the computational basis as

$$\tilde{P} = \sum_{i,\alpha,\beta} A^i_{\alpha\beta} |i\rangle\langle\alpha,\beta| \ .$$

   Now consider the map $\tilde{Q} = (\tilde{P}_{AB} \otimes \tilde{P}_{CD})|\omega\rangle_{BC}$ (cf. Problem 11.1, Eq. (1)). What is the explicit form of

$$\tilde{Q} = \sum_{i,j,\alpha,\gamma} B^{i,j}_{\alpha\gamma} |i,j\rangle\langle\alpha,\gamma|$$

   in terms of the $A^i_{\alpha,\beta}$?

4. Iterate the previous formula to obtain an explicit expression of the state $|\Psi\rangle$ in terms of the tensor $A^i_{\alpha,\beta}$.

5. Determine the explicit form of the $A^i_{\alpha,\beta}$ for the AKLT state.

6. Show that for the AKLT model, there exists a physical basis transformation such that the A^i become (proportional to) the Pauli matrices. What are the spin operators $\{S^1_x, S^1_y, S^1_z\}$ in this basis?