II. Entanglement

1. Mean field theory – and beyond

Aim: Find & describe (in a useful way) ground state $|\psi_0\rangle$ of quantum many-body (QMB) system with local interactions:

$$H = \sum h_i \; ; \qquad H|\psi_0\rangle = E_0|\psi_0\rangle$$

a) Variational wavefunctions

"Variational principle":

G.S. $|\psi_0\rangle$ is vector $|\psi\rangle \in \mathcal{H} = (\mathbb{C}^d)^{\otimes N}$

which obtains minimum

$$\min_{|\psi\rangle} \langle\psi|H|\psi\rangle = E_0$$

(E.g. follows (in finite dim.) from eigenvalue decomposition $H = \sum E_i |\psi_i\rangle\langle\psi_i|$; $E_0 \le E_1 \le \dots$:

$$\langle\psi|H|\psi\rangle = \sum |\langle\psi|\psi_i\rangle|^2 E_i = \sum p_i \underbrace{E_i}_{\geq E_0} \geq E_0 \quad)$$

Use a (educated) guess for the form of $|\psi_0\rangle$:

$$|\psi_0\rangle \approx |\tilde{\psi}_0\rangle \in \mathcal{S} \subset (\mathbb{C}^d)^{\otimes N}$$

"variational family of states"

& minimize $\langle\psi|H|\psi\rangle$ over $|\psi\rangle \in \mathcal{S}$.

We need/want a good family $\mathcal{F}$, i.e. one which

- approximates ground state well
- is simple & useful to work with

b) Mean-field theory

Simplest guess for $\mathcal{F}$: Ignore (quantum) correlations:

Mean-field ansatz:

$$\mathcal{F}_{MF} = \{ |\phi_1\rangle \otimes |\phi_2\rangle \otimes \dots \otimes |\phi_N\rangle \,,\ |\phi_i\rangle \in \mathbb{C}^d \}$$

→ works surprisingly well in many cases - especially in higher spatial dimensions (Reason: "monogamy of entanglement" - if q. correlations are shared betw. many neighbors, they must be small, cf. later.)

Example I:

Ising model (in D dims.), $H = -\sum_{\langle ij\rangle} Z_i Z_j - h \sum X_i$

nearest neighbors

→ Homework suggestion

Example II:

Heisenberg antiferromagnet in 1D

(PBC = periodic boundary conditions)

$$H = \sum_{i=1}^{N} \vec{S}_i \cdot \vec{S}_{i+1} \quad , \; N \text{ even}$$

$$\vec{S}_i \cdot \vec{S}_{i+1} = \frac{1}{4} \begin{pmatrix} 1 & & & \\ & -1 & 2 & \\ & 2 & -1 & \\ & & & 1 \end{pmatrix}$$

$$\left(\langle \phi_1 | \otimes \langle \phi_2 | \right) \left(\vec{S}_1 \otimes \vec{S}_2 \right) \left(|\phi_1\rangle \otimes |\phi_2\rangle \right) = \, ?$$

- use rotational invariance of $\vec{S}_1 \cdot \vec{S}_2$:

  can fix wlog $|\phi_1\rangle = |0\rangle$.

- then $\langle \phi_1 | \, \vec{S}_1 \cdot \vec{S}_2 \, | \phi_1 \rangle$ $\equiv |\phi_1\rangle\langle\phi_1| \otimes \mathbb{1}_2$: only project spin 1

$$= \langle 0 | \vec{S}_1 \cdot \vec{S}_2 | 0 \rangle$$

$$= \frac{1}{4} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$\Rightarrow$ optimal value for $|\phi_2\rangle = |1\rangle$:

$$\left(\langle \phi_1 | \otimes \langle \phi_2 | \right) \left(\vec{S}_1 \cdot \vec{S}_2 \right) \left(|\phi_1\rangle \otimes |\phi_2\rangle \right) = -\frac{1}{4}$$

- continue sequentially:

$$|\tilde{\psi}_0\rangle = |0\rangle \otimes |1\rangle \otimes |0\rangle \otimes |1\rangle \otimes \dots \otimes |0\rangle \otimes |1\rangle$$

(or any rotated version, $U^{\otimes N} |\tilde{\psi}_0\rangle$)

$\Rightarrow$ antiferromagnetic order

- energy per site from mean-field:

$$\frac{E_0}{N} = -\frac{1}{4}.$$

c) Beyond mean field

How good is this?

Have seen: $\mathrm{eig}(\vec{S}_1 \cdot \vec{S}_2) = \{-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\}$.

$\Rightarrow$ Theoretical lower bound $\frac{E_0}{N} \geq -\frac{3}{4}$!

How close can we get to that?

<u>Improved guess:</u>

For a <u>single</u> pair $\vec{S}_i \cdot \vec{S}_{i+1}$, the singlet state $|\sigma_{i,i+1}\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$ has optimal energy $-\frac{3}{4}$.

<u>Ansatz:</u> $|\tilde{\psi}_0\rangle \overset{?}{=} |\sigma_{12}\rangle \otimes |\sigma_{34}\rangle \otimes \dots \otimes |\sigma_{N-1\,N}\rangle$

What is energy $\langle \tilde{\psi}_0 | H | \tilde{\psi}_0 \rangle$?

$$\langle \tilde{\psi}_0 | \vec{S}_1 \cdot \vec{S}_2 | \tilde{\psi}_0 \rangle = \langle \sigma_{12} | \vec{S}_1 \cdot \vec{S}_2 | \sigma_{12} \rangle = -\frac{3}{4}$$

$$\langle \tilde{\psi}_0 | \vec{S}_2 \cdot \vec{S}_3 | \tilde{\psi}_0 \rangle =$$

$$= \sum_\alpha \left(\langle \sigma_{12} | \otimes \langle \sigma_{34} | \right) \left(S_2^\alpha \cdot S_3^\alpha \right) \left(|\sigma_{12}\rangle \otimes |\sigma_{34}\rangle \right)$$

and $\langle \sigma_{12} | S_2^\alpha | \sigma_{12} \rangle = 0$:

direct verification, or (cf. later) red. density matrix of $|\sigma_{12}\rangle$.

$$\dots = 0 .$$

$$\Rightarrow \frac{E_0}{N} = -\frac{1}{2} \cdot \frac{3}{4} = -\frac{3}{8} < -\frac{1}{4} \; !$$

__Better energy__ If we include q. correlations.

Intuition: Hamiltonian $H = \sum h_i$ has interactions h_i whose ground states have q. correlations – entanglement.

However, we have only included correlations between __half the pairs__! Ideally, we would like to have __all__ nearest neighbor pairs in the state $|\sigma_{i,i+1}\rangle$. Is this possible?

No! "__Monogamy of entanglement__"

A spin cannot be maximally entangled with several other spins.

If entanglement with several other spins is required, the entanglement has to be split up between the partners.

($\rightarrow$ in higher dimensions, there are more neighbors: entanglement between each pair smaller, and thus, mean field works better.)

<u>In particular:</u> Impossible to get $\frac{E_0}{N} = -\frac{3}{4}$.

<u>But:</u> True value is known to be

$$\frac{E_0}{N} \xrightarrow{N\to\infty} \frac{1}{4} - \log 2 \approx -0.443 < -\frac{3}{8}$$

$\Rightarrow$ In order to find a variational family which allows us to approx. this value, we need to entangle all adjacent spins (in order to minimize $\langle \tilde{\varphi}_0 | H | \tilde{\varphi}_0 \rangle = \sum \langle \tilde{\varphi}_0 | h_i | \tilde{\varphi}_0 \rangle$)

$\Rightarrow$ need to understand entanglement structure of ground states of local interactions across any cut.

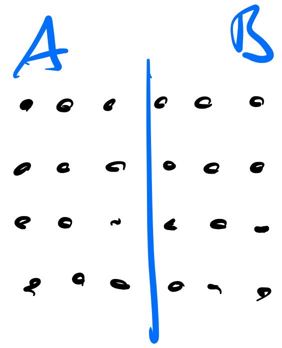
2. The Schmidt decomposition

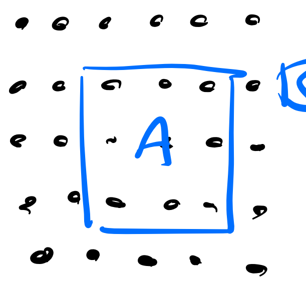
a) Setup

Consider a system consisting of two parts:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B .$$

(In a many-body system, this could come from a bipartition:





)

- States $|\psi\rangle$ which can be written as

$$|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$$

$$\left(\text{i.e.: } |\psi_A\rangle = \sum a_i |i\rangle,\ |\psi_B\rangle = \sum b_j |j\rangle : \right.$$

$$\left. |\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle = \sum a_i b_j |i,j\rangle \right).$$

for some $|\psi_A\rangle, |\psi_B\rangle$ are called **product states** or **separable states**.

- States $|\psi\rangle$ which are <u>not</u> of this form, i.e. which <u>cannot</u> be written as $|\psi_A\rangle \otimes |\psi_B\rangle$, are called <u>entangled</u>.

That is,

$$(*) \quad |\psi\rangle = \gamma_1 |\psi_{A,1}\rangle \otimes |\psi_{B,1}\rangle + \gamma_2 |\psi_{A,2}\rangle \otimes |\psi_{B,2}\rangle + \ldots$$

with <u>more</u> than one term!

This suggests that entanglement is related to some kind of correlations betw. A & B:

$$|\psi_{A,1}\rangle \longleftrightarrow |\psi_{B,1}\rangle \qquad (\text{weight } |\gamma_1|^2)$$

$$|\psi_{A,2}\rangle \longleftrightarrow |\psi_{B,2}\rangle \qquad (\text{weight } |\gamma_2|^2)$$

$\vdots$ etc.

How to <u>characterize the entanglement</u>?

Intuitively, it should depend on weights $|\gamma_k|^2$

and distinguishability $1-|\langle\psi_{A,k}|\psi_{A,\ell}\rangle|^2$ &

$1-|\langle\psi_{B,k}|\psi_{B,\ell}\rangle|^2$.

But naively, this is not even invariant under writing $|\psi\rangle$ in different ways as $(*)$.

Q: How can we characterize entanglement in a meaningful way?

6) The singular value decomposition

Theorem (Singular Value Decomposition, SVD):

Any complex $m\times n$-matrix M can be written as

$$M = UDV^\dagger,$$

with U,V isometries (i.e. $U^\dagger U = V^\dagger V = I$), and

$$D = \begin{pmatrix} s_1 & & & 0 \\ & s_2 & & \\ & & \ddots & \\ 0 & & & s_r \end{pmatrix} ; \quad r \le m,n.$$

with $s_1 \geq s_2 \geq \dots \geq s_r > 0$ the singular values of M.

The s_k are the non-zero eigenvalues of $MM^\dagger$ or equivalently of $M^\dagger M$.

(Note: U, V are unique up to rotations in subspaces of degen. s_i. Often, the SVD is stated with U, V unitary and D a $n \times m$-matrix. It is obtained from the form above by padding D with zeros and completing U and V to unitaries by adding columns.)

Proof: Diagonalize $MM^\dagger$:

$$MM^\dagger = W \Lambda W^\dagger; \quad W \text{ unitary},$$

$$\Lambda = m\left\{\underbrace{\begin{pmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \ddots & & & \\ & & & \lambda_r & & \\ & & & & 0 & \\ & & & & & \ddots \; 0 \end{pmatrix}}_{m}\right. \quad \text{with } \lambda_1 \geq \lambda_2 \geq \dots \lambda_r > 0$$

$$\text{define } \Pi := \underbrace{\left(\begin{array}{ccc|c} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \end{array}\right)}_{m}\Big\} r,$$

$$U := W\Pi^\dagger, \quad D := \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_r} \end{pmatrix}, \quad \text{and}$$

$$V^\dagger := D^{-1}\Pi W^\dagger M.$$

Then, $U^\dagger U = \Pi W^\dagger W \Pi^\dagger = \Pi \cdot I \cdot \Pi^\dagger = I$,

$$V^\dagger V = D^{-1}\underbrace{\Pi \underbrace{W^\dagger M M^\dagger W}_{=\Lambda} \Pi^\dagger}_{=D^2} D^{-1} = I,$$

(i.e. U, V isometries), and

$$\underbrace{(I - \Pi^\dagger\Pi)}_{= \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & 1 \ddots 1 \end{pmatrix}} W^\dagger M M^\dagger W (I - \Pi^\dagger\Pi) = (I - \Pi^\dagger\Pi)\Lambda(I - \Pi^\dagger\Pi) = 0,$$

$$\Longrightarrow (I - \Pi^\dagger\Pi) W^\dagger M = 0. \quad \text{Thus,}$$

$$\underline{\underline{UDV^\dagger}} = (W\Pi^\dagger) D (D^{-1}\Pi W^\dagger M)$$

$$= W\Pi^\dagger\Pi W^\dagger M = W I W^\dagger M = \underline{\underline{M}}.$$

c) The Schmidt decomposition

Back to bipartite state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.

Consider ONB's $|i\rangle_A$, $|j\rangle_B$.

Write

$$|\psi\rangle = \sum c_{ij} |i\rangle_A |j\rangle_B .$$

Use SVD $C = (c_{ij}) = U \cdot D \cdot V^\dagger$,

i.e. $c_{ij} = \sum u_{ik} s_k \overline{v_{jk}}$

$$\Rightarrow |\psi\rangle = \sum_k s_k \underbrace{\Big(\sum_i u_{ik} |i\rangle\Big)_A}_{=: |\psi_A^k\rangle} \underbrace{\Big(\sum_j \overline{v_{jk}} |j\rangle\Big)}_{= |\psi_B^k\rangle}$$

$=: |\psi_A^k\rangle$ ONS as u_{ik} isometry!

$= |\psi_B^k\rangle$ ONS as $\overline{v_{jk}}$ isometry!

$$\Rightarrow \boxed{|\psi\rangle = \sum_{k=1}^{r} s_k |\psi_A^k\rangle \otimes |\psi_B^k\rangle}$$

with $s_k > 0$.

The Schmidt decomposition, with Schmidt coefficients $s_k > 0$, and Schmidt rank r.

Note: • The $\{|\psi_A^k\rangle\}$ and the $\{|\psi_B^k\rangle\}$ each form an orthonormal set!

• The Schmidt decomposition is unique, up to simultaneous rotations within subspaces w/ degenerate Schmidt coefficients.

d) Reduced density matrices

Density matrices: typ. introduced to describe states where we have partial knowledge:

Consider $\langle\psi|\Pi|\psi\rangle$, with Π e.g. an observable, or a projection onto a meas. result:

$$\langle\psi|\Pi|\psi\rangle = \mathrm{tr}[\Pi \cdot \underbrace{|\psi\rangle\langle\psi|}_{\text{Projector onto } |\psi\rangle}]$$

$\mathrm{tr}(X) = \sum X_{ii}$. Basis-independent!

Then, if we have state $|\psi_i\rangle$ w/ probability p_i:

Avg. outcome is

$$\sum p_i \langle\psi_i|\Pi|\psi_i\rangle = \sum p_i \,\mathrm{tr}[\Pi|\psi_i\rangle\langle\psi_i|]$$

$$= \mathrm{tr}[\Pi \cdot \sum p_i|\psi_i\rangle\langle\psi_i|]$$

$$= \mathrm{tr}[\Pi\rho]$$

with $\rho := \sum p_i|\psi_i\rangle\langle\psi_i|$ the density matrix (or density operator).

(Can be used to describe ensemble $\{p_i, |\psi_i\rangle\}$.

Note: This is not uniquely determined by ρ!)

Back to bipartite states. Consider $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.

How can we describe the expectation value of an operator Π_A on A? (E.g. measurement)

Operator "ignores" B system. Thus, on any product state $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$, we must have

$$\text{"}\langle\psi|\Pi_A|\psi\rangle\text{"} := \langle\psi_A|\Pi_A|\psi_A\rangle$$

$$= \langle\psi_A|\Pi_A|\psi_A\rangle \langle\psi_B|\psi_B\rangle$$

That is, Π_A acts on $|\psi_A\rangle \otimes |\psi_B\rangle$ as

$$|\psi_A\rangle \otimes |\psi_B\rangle \longmapsto \left(\Pi_A|\psi_A\rangle\right) \otimes |\psi_B\rangle.$$

This is exactly the definition of the operator $\underline{\underline{\Pi_A \otimes \mathbb{1}_B}}$! — Due to linearity, Π_A must act as $\Pi_A \otimes \mathbb{1}_B$ on <u>all</u> states $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$!

Now let $|\psi\rangle = \sum c_{ij}\, |i\rangle_A \otimes |j\rangle_B$.

Then, $\langle\psi|\, \Pi_A \otimes \mathbb{1}_B\, |\psi\rangle =$

$$= \sum c_{ij}\, \overline{c_{i'j'}}\, \left(\langle i'|_A \otimes \langle j'|_B\right) \left(\Pi_A \otimes \mathbb{1}_B\right) \left(|i\rangle_A \otimes |j\rangle_B\right)$$

$$= \sum c_{ij} \overline{c_{i'j'}} \underbrace{\langle i' | \Pi_A | i \rangle}_{= tr[\Pi_A |i\rangle\langle i'|]} \underbrace{\langle j' | \mathbb{1}_B | j \rangle}_{= \delta_{jj'}}$$

$$= \sum_{ii'j} c_{ij} \overline{c_{i'j}} \, tr\left[\Pi_A |i\rangle\langle i'|\right]$$

$$= tr\left[\Pi_A \rho\right],$$

with $\rho = \sum_{ii'j} c_{ij} \overline{c_{i'j}} |i\rangle\langle i'|$,

or $\rho_{ii'} = \langle i | \rho | i' \rangle = (CC^\dagger)_{ii'}$, $C = (c_{ij})$.

This can be formalized through the concept of the partial trace: Given ρ_{AB}, the partial trace is

$$\rho_A = tr_B \, \rho_{AB} := \sum_j \left(\mathbb{1}_A \otimes \langle j|_B\right) \rho_{AB} \left(\mathbb{1}_A \otimes |j\rangle_B\right)$$

$$\equiv \sum_{j'} \langle j'_B | \rho_{AB} | j' \rangle_B$$

$$\equiv \sum_{i, i', j} |i\rangle_A \langle i, j | \rho_{AB} | i', j \rangle \langle i' |_A$$

Again, ρ_A describes anything pertaining to system A.

In particular, for the case $\rho_{AB} = |\psi\rangle\langle\psi|$,

$|\psi\rangle = \sum c_{ij} |i\rangle \otimes |j\rangle$:

$$\rho_A = \sum c_{ij} \overline{c_{i'j'}} \, \mathrm{tr}_B \left[\left(|i\rangle_A \langle i'|_A \right) \otimes |j\rangle_B \langle j'|_B \right]$$

$$= \sum c_{ij} \overline{c_{i'j'}} \, |i\rangle_A \langle i'|_A \underbrace{\mathrm{tr} \left[|j\rangle\langle j'| \right]}_{= \delta_{jj'}.}$$

Finally, consider Schmidt decomposition of $|\psi\rangle$:

$$|\psi\rangle = \sum_{k=1}^{r} s_k |\psi_A^k\rangle \otimes |\psi_B^k\rangle.$$

Then,

$$\underline{\rho_A} = \mathrm{tr}_B \left[\sum_{k,l} s_k s_l \, |\psi_A^k\rangle\langle\psi_A^l| \otimes |\psi_B^k\rangle\langle\psi_B^l| \right]$$

$$= \sum_{kl} s_k s_l \, |\psi_A^k\rangle\langle\psi_A^l| \, \underbrace{\mathrm{tr}_B\left[|\psi_B^k\rangle\langle\psi_B^l|\right]}_{}$$

$= \delta_{kl}$ as $|\psi_B^\bullet\rangle$ ONS
(cyclicity or trace in $|\psi_B^\bullet\rangle$)

$$\underline{= \sum_k s_k^2 |\psi_A^k\rangle\langle\psi_A^k| .}$$

Similarly, $\rho_B = \mathrm{tr}_A |\psi\rangle\langle\psi| = \sum_k s_k^2 |\psi_B^k\rangle\langle\psi_B^k|$.

$\Rightarrow$ <u>Schmidt coefficients</u> are the <u>non-zero</u>
<u>eigenvalues of ρ_A (or ρ_B)</u>.

(In particular: For a pure state $|\psi\rangle = |\psi\rangle_{AB}$,
ρ_A and ρ_B have the same non-zero eigenvals.)

The <u>Schmidt vectors</u> are the eigenvectors of
ρ_A & ρ_B, respectively.

Unless there are degenerate s_k, this uniquely determines the Schmidt decomposition.

e) Quantitative characterization of entanglement

Recall: $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$:

$$|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \quad :\Longleftrightarrow \quad |\psi\rangle \text{ product (or separable)}$$

$$|\psi\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle \quad \Longleftrightarrow \quad |\psi\rangle \text{ entangled}$$

– i.e., $|\psi\rangle$ has non-trivial quantum correlations which cannot be created by local operations & classical communication.

What determines if, and how much, a state is entangled?

Use Schmidt basis:

$$|\psi\rangle = \sum s_k |\psi_A^k\rangle \otimes |\psi_B^k\rangle$$

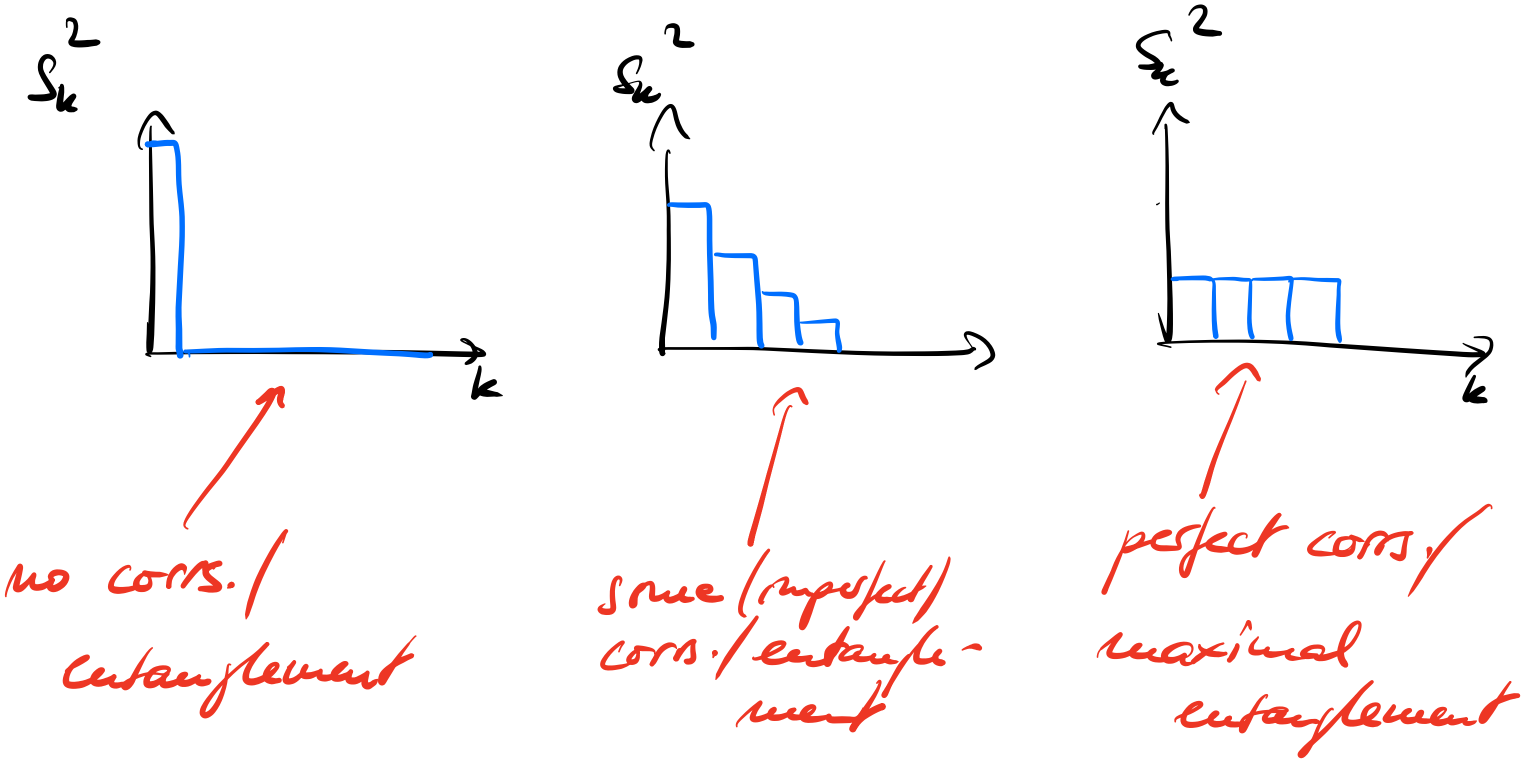
$|\psi_A^k\rangle$ ONS, $|\psi_B^k\rangle$ ONS: For each k, we have perfect (i.e. orthogonal/distinguishable) correlations between A & B. The nature (& amount) of correlations should dep. on the distribution of the s_k — if more events can occur with same probability, there are more correlation.

Indeed: the $|\psi_A^k\rangle$ & $|\psi_B^k\rangle$ can be changed with local rotations, and this is all that local rotations can do $\Rightarrow$ all info. about entanglement is in the s_k.

Conversely, the s_k cannot be changed by local unitaries, as they are eigenvalues of ρ_A & ρ_B (which are unchanged under U_B / U_A, respectively).

(Alternatively: local unitary transforms $C \to U_A C U_B^T$, which does not change singular values of C.)

Resulting picture:



(Intuitively:)

Amount of entanglement $\longleftrightarrow$ Amount of disorder in $p_k = s_k^2$.

$\left(\sum p_k = 1 \,!\right)$

Typically measured by some measure of entropy ("entanglement entropy"),

e.g. von Neumann entropy

$$S(\rho_A) := -\mathrm{tr}\left[\rho_A \log \rho_A\right] = -\sum p_k \log p_k$$

Defined on the eigenvalues, i.e.

$$\rho_A = \sum p_i |\psi_i\rangle\langle\psi_i| \Rightarrow \log \rho_A = \sum \log p_i |\psi_i\rangle\langle\psi_i|$$

$$\Rightarrow S(\rho_A) = -\mathrm{tr}\left[\sum p_i \log p_i \, |\psi_i\rangle\langle\psi_i|\right]$$

$$= -\sum p_i \log p_i$$

or some <u>Renyi entropy</u>

$$S_\alpha(\rho_A) = \frac{1}{1-\alpha} \log\left(\mathrm{tr}(\rho^\alpha)\right) = \frac{1}{1-\alpha} \log \sum p_k^\alpha .$$

(Note: $\lim_{\alpha \to 1} S_\alpha(\rho) = S(\rho)$).

f) <u>Approximation</u> by <u>low Schmidt rank</u>

$$|\psi\rangle = \sum_{k=1}^{r} s_k |\psi_A^k\rangle \otimes |\psi_B^k\rangle \quad \text{Schmidt dec.,}$$

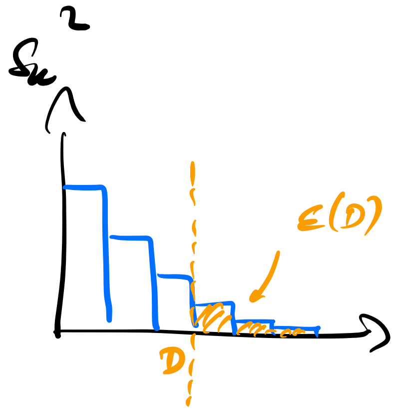
$$s_1 \geq s_2 \geq \dots > 0 .$$

We can <u>approximate</u> $|\psi\rangle$ by a state of Schmidt rank D by cutting the sum,

$$|\Phi_D\rangle := \sum_{k=1}^{D} s_k \, |\psi_A^k\rangle \otimes |\psi_B^k\rangle .$$

This is the <u>optimal</u> truncation, i.e. the one which maximizes $|\langle\psi|\Phi_D\rangle|$. The truncation error is

$$\epsilon(D) = 1 - |\langle\psi|\Phi_D\rangle| = \sum_{k>D} s_k .$$



If the s_k^2 decay rapidly enough (small weight in tail), then the <u>error is small.</u>

In particular, this is the case if the Renyi-entropies for some $\alpha < 1$ are bounded. Then,

$$\epsilon(D) \leq \frac{1}{D^{\eta_\alpha}} \, C_\alpha \, e^{\eta_\alpha S_\alpha(\rho_A)}$$

with $C_\alpha = \frac{1}{2}\alpha(1-\alpha)^{\eta_\alpha}$; $\eta_\alpha = \frac{1-\alpha}{\alpha}$

(See https://arxiv.org/abs/cond-mat/0505140)

– i.e., the error scales as $\varepsilon(D) \sim 1/\mathrm{poly}(D)$.

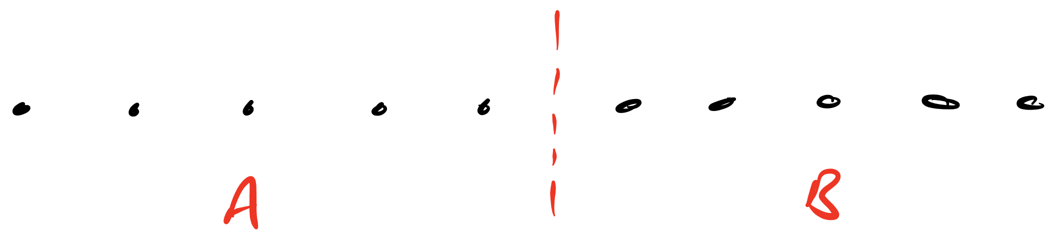
g) Entanglement in ground states

Ground states of quantum spin systems aim to minimize $\sum \langle \psi_0 | h_i | \psi_0 \rangle$ $\Rightarrow$ intuitively, q. correlations are built up locally.

This is captured by the area law for entanglement:

The entanglement across every cut scales like the length of the boundary (vs. the volume of the region).

E.g. 1D chain: $|\psi_0\rangle$



$$\rho_A = \mathrm{tr}_B |\psi_0\rangle\langle\psi_0|$$

$S(\rho_A) \leq \text{const}$ (also holds for Renyi entropies)

Rigorously proven for <u>1D gapped systems.</u>

<u>proven</u> by Hastings https://arxiv.org/abs/0705.2024

improved by Arad, Kitaev, Landau, Vazirani https://arxiv.org/abs/1301.1162

For gapless systems in 1D:

$$S(\rho_A) \sim \log(\underbrace{|A|}_{\text{size of } A})$$

for physically reasonable cases, but (artificial) counterexamples exist.

<u>2D</u>: spin systems (gapped & gapless):

$$S(\rho_A) \sim \underbrace{|\partial A|}_{\text{length of boundary}}$$

gapless fermions (= metals):

$$S(\rho_A) \sim |\partial A| \log |A|$$

- all of this not proven, but believed to hold for reasonable systems.

Thus: Even critical systems generally display only a logarithmic entanglement scaling.

This is in stark contrast to a random (Haar-random) state, for which

$$S(\rho_A) = |A| - c \log |A| \,!$$

$\Rightarrow$ ground states are very special in the space of all states!

(We knew this from parameter counting, but now we know what makes them special: They have (comparatively) very little entanglement!)

So... what is the structure of many-body states with little entanglement?