V. Solvable models and the classification of phases

1. The AKLT model

a) Construction

Let $|\omega\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right) \in \mathbb{C}^2 \otimes \mathbb{C}^2$

$|\omega\rangle$ is $SU(2)$-invariant:

$$(u \otimes u)|\omega\rangle = |\omega\rangle \qquad \forall u \in SU(2)$$

The space $\mathbb{C}^2 \otimes \mathbb{C}^2$ can be decomposed naturally into a anti-sym. and sym. space,

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathcal{A} \oplus \mathcal{S},$$

with $\mathcal{S} = \operatorname{span}\{ |S=1, m=+1\rangle, |S=1, m=0\rangle, |S=1, m=1\rangle \}$,

where
$$|S=1, m=+1\rangle = |00\rangle$$
$$|S=1, m=0\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$$
$$|S=1, m=-1\rangle = |11\rangle$$

and $A = \text{span}\ \{ |S=0, m=0\rangle \}$

$$|S=0, m=0\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle \right)$$

Thus naturally decomposes

$$\underbrace{u \otimes u \cong \mathbb{1}}_{\text{spin-0 space}} \oplus \underbrace{V_u}_{\text{spin-1 space \& action}}$$

Now define

$$P : \mathbb{C}^2 \otimes \mathbb{C}^2 \longrightarrow \mathbb{C}^3$$

$$P = |+1\rangle\langle S=1, m=+1| + |0\rangle\langle S=1, m=0| + |-1\rangle\langle S=1, m=-1|$$

the isometry projecting onto the spin-1 space.

In particular,

$$P(u \otimes u) = V_u P.$$

(Note: Up to a phase ± 1, $u \in SU(2)$ can be understood as the rotation of a spin-$\frac{1}{2}$ particle by an angle of $|\vec{\theta}|$ about an axis $\vec{n} = \vec{\theta}/|\vec{\theta}|$ (i.e., $\vec{\theta} \in so(3)$):

$$u = u(\vec{\theta}) = e^{i\vec{\theta}\cdot\vec{S}^{1/2}},$$

with $\vec{S}^{1/2} = (\frac{1}{2}\sigma_x, \frac{1}{2}\sigma_y, \frac{1}{2}\sigma_z)$ the spin-1/2 operators.

Then, similarly, $V_{\vec{\theta}} = V_{u(\vec{\theta})} = e^{i\vec{\theta}\cdot\vec{S}^1}$,
with $\vec{S}^1 = (S_x^1, S_y^1, S_z^1)$ the spin-1 operators.)

Now consider a chain of $2N$ spins $\mathbb{C}^2$, and construct the state

$$|\Psi_{AKLT}\rangle = \big(P_{12} \otimes P_{34} \otimes \ldots \otimes P_{N-1,N}\big)\big(|\omega\rangle_{23} \otimes |\omega\rangle_{45} \otimes \ldots$$
$$\ldots \otimes |\omega\rangle_{N-2,N-1} \otimes |\omega\rangle_{N,1}\big)$$

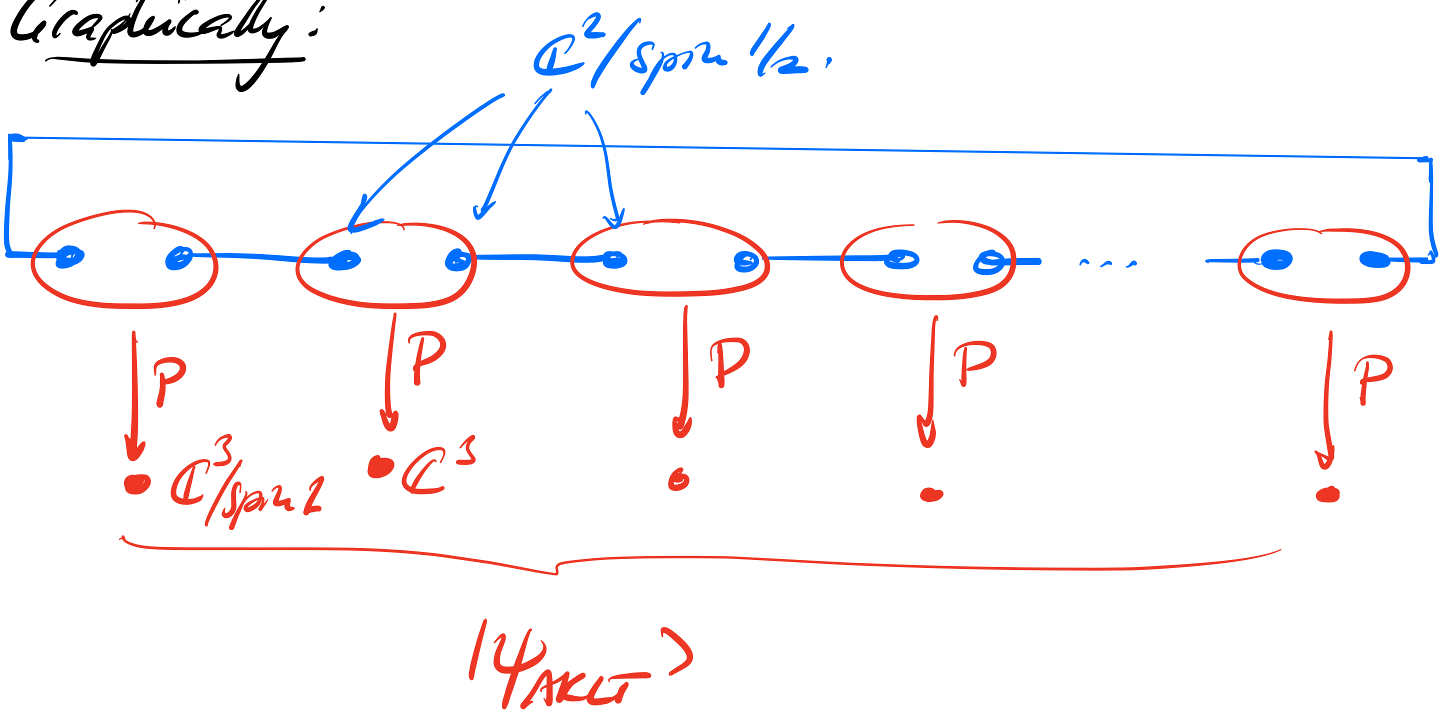
spins on which it acts

$$= \text{“}\left(P^{\otimes N}\right)\left(|\omega\rangle^{\otimes N}\right)\text{”}.$$

This is the <u>AKLT state</u>.

(Affleck, Kennedy, Lieb, Tasaki)

<u>Graphically</u>:



The AKLT state is rotationally (i.e. SO(3)) invariant:

$$\underline{V_{\vec{\theta}}^{\otimes N}\,|\psi_{AKLT}\rangle} = V_{\vec{\theta}}^{\otimes N}\left(P^{\otimes N}\right)\left(|\omega\rangle^{\otimes N}\right)$$

$$= \left(P\left(u_{\vec{\theta}}\otimes u_{\vec{\theta}}\right)\right)^{\otimes N}\,|\omega\rangle^{\otimes N}$$

$$= P^{\otimes N}\,u_{\vec{\theta}}^{\otimes 2N}\,|\omega\rangle^{\otimes N}$$

$$= P^{\otimes N} \Big(\underbrace{(u_\theta \otimes u_\theta)\,|\omega\rangle}_{=|\omega\rangle} \Big)^{\otimes N}$$

$$= P^{\otimes N} |\omega\rangle^{\otimes N} = \underline{|\psi_{AKLT}\rangle}$$

Key point: This was achieved by encoding the symmetry locally in the building blocks!

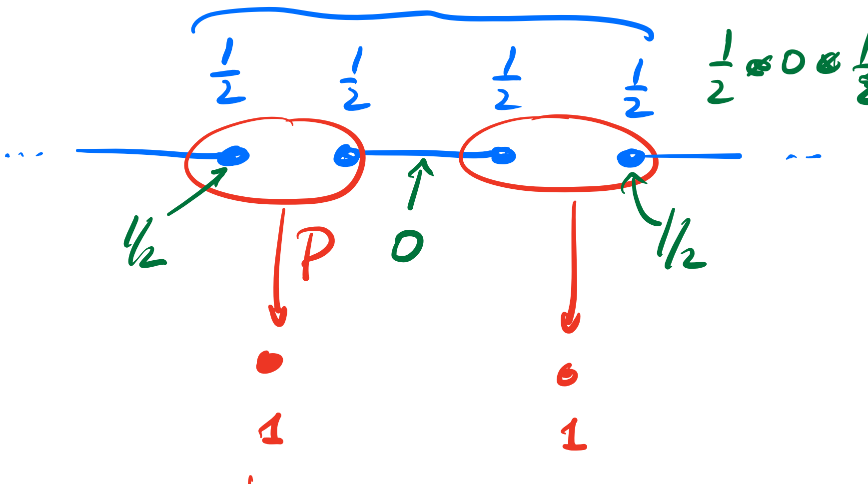
b) Parent Hamiltonians

Does the AKLT model appear as ground state of some local Hamiltonian?

Consider 2 consecutive sites in the AKLT chain,

$$\tfrac12 \otimes \tfrac12 \otimes \tfrac12 \otimes \tfrac12 = 0 \oplus 1 \oplus 2$$

$$\tfrac12 \otimes 0 \otimes \tfrac12 = 0 \oplus 1$$



$$1 \otimes 1 = 0 \oplus 1 \cancel{\oplus 2}$$

Auxiliary spin $\frac{1}{2}$ - deg. of freedom are in spin $\frac{1}{2}$, 0, $\frac{1}{2}$, respectively,

and P does not change spin:

$\Rightarrow$ final state on 2 sites (i.e., reduced density matrix after proj. out the rest of the chain) cannot have spin 2!

i.e.: $\quad tr_{3,4,\ldots,N}\left(|\psi_{AKLT}\rangle\langle\psi_{AKLT}|\right) = \rho_{12}$

is supported on space with spin $= 0, 1$.

(i.e.: $\mathbb{C}^3 \otimes \mathbb{C}^3 = \mathcal{H}_{S=0} \oplus \mathcal{H}_{S=1} \oplus \mathcal{H}_{S=2}$)

and $\mathrm{supp}\,(\rho_{12}) \subset \mathcal{H}_{S=0} \oplus \mathcal{H}_{S=1}$.)

Define Hamiltonian acting on sites 1,2:

$$h_{12} = \Pi_{S=2},$$

where $\Pi_{S=2}$ is the projector onto the spin-2 space (i.e. $\Pi_{S=2}$ projects onto $\mathcal{H}_{S=2}$).

Then, $\langle \psi_{AKLT} | h_{12} | \psi_{AKLT} \rangle$

$$= tr[\rho_{12} h_{12}] = 0.$$

Since $eig(h_{12}) = \{0,1\}$, this means $|\psi_{AKLT}\rangle$ is an eigenstate of h_{12} with eigenvalue 0:

$$h_{12} |\psi_{AKLT}\rangle = 0.$$

We can use the same argument for any 2 adjacent sites $i, i+1$:

$$h_{i,i+1} = \underbrace{(\Pi_{S=2})_{i,i+1}}_{\text{proj. on } S=2 \text{ subspace at sites } i,i+1.}$$

$$h_{i,i+1} |\psi_{AKLT}\rangle = 0.$$

Then, for $H_{AKLT} = \sum_{i=1}^{N} h_{i,i+1}$, we have:

$$H_{AKLT} = \sum_{i=1}^{N} \underbrace{h_{i,i+1}}_{\geq 0} \geq 0$$

(i.e. H only has eigenvalues ≥ 0), and

$$H_{AKLT} |\psi_{AKLT}\rangle = \sum_{i=1}^{N} \underbrace{h_{i,i+1} |\psi_{AKLT}\rangle}_{=0} = 0$$

$\Longrightarrow$ $|\psi_{AKLT}\rangle$ is a ground state of H_{AKLT}!

Note: A Hamiltonian $H = \sum h_{i,i+1}$ where the ground state minimizes the energy of each $h_{i,i+1}$ individually is called frustration free.

Moreover, H_{AKLT} is rotationally invariant:

$$(V_{\vec{\theta}} \otimes V_{\vec{\theta}})\, h_{12} = h_{12} (V_{\vec{\theta}} \otimes V_{\vec{\theta}}),$$

since a subspace of constant spin is invariant under rotations (but not the vectors in the space!).

Note:

$$H_{AKLT} = \sum_{i=1}^{N} \left(\frac{1}{2}\left(\vec{S}_i \cdot \vec{S}_{i+1}\right) + \frac{1}{6}\left(\vec{S}_i \cdot \vec{S}_{i+1}\right)^2 + \frac{1}{3} \right).$$

One can show:

Theorem: $|\psi_{AKLT}\rangle$ is the unique ground state of H_{AKLT}.

Theorem H_{AKLT} has a gap.

(We will not prove this here – possibly in 2nd part of the lecture.)

c) The Haldane conjecture

Consider spin - S Heisenberg model:

$$H = \sum_{i=1}^{N} \vec{S}_i \cdot \vec{S}_{i+1}$$

$S = 1/2, 3/2, 5/2, \dots$:

Lieb-Schultz-Mattis-Theorem ('61):

H is symmetry breaking or gapless — that is, H cannot be gapped with a unique ground state.

$S = 0, 1, 2, \dots$:

Haldane ('83): $H = \sum \vec{S}_i \cdot \vec{S}_{i+1}$ for integer spin has unique ground state + gap.

Argument works via mapping to field theory ("non-linear sigma model") in limit $S \to \infty$ $\to$ not fully rigorous.

Argument based on symmetries of model.

<u>AKLT model:</u> Provides a rigorous variant of an integer spin chain with SU(2) symmetry where the gap can be <u>rigorously prove</u>!

d) Fractional edge modes

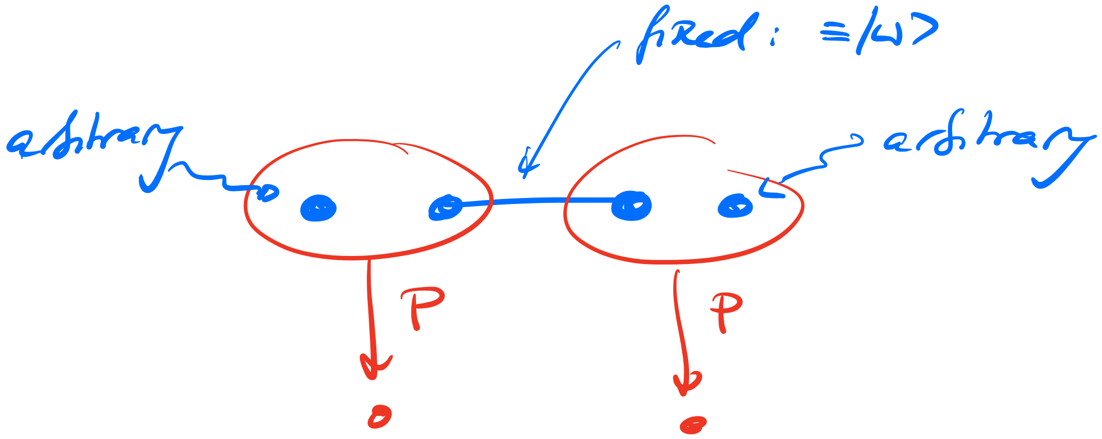
Consider AKLT Hamiltonian w/ open boundaries:

$$H = \sum_{i=1}^{N-1} h_{i,i+1}$$

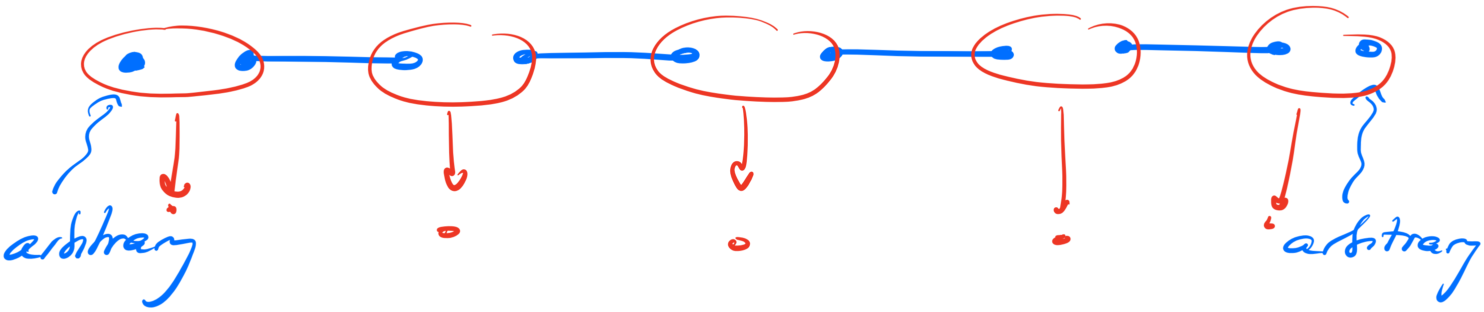
What are the ground states?

Intuition (made rigorous in proof of uniqueness of g.s.):

Each $h_{i,i+1}$ enforces that on sites $i, i+1$, the ground state looks like



On OBC: Precisely spins on edge remain arbitrary.



→ AKLT chain w/ OBC has 4-fold degen. ground space.

→ Parametrized by a spin-1/2 degree of freedom ("edge mode") at each boundary.

→ Edge modes localized at boundaries (→ more later) → each edge carries a spin-1/2 excitation.

This is <u>very surprising</u>: In a spin system, local excitations – created e.g. by S^+ – should have <u>integer spin</u> (as S^+ changes spin by 1).

⟹ The spins "fractionalize" at the edge; and such fractional excitations can only be created <u>in pairs</u>.

Unconventional behavior:

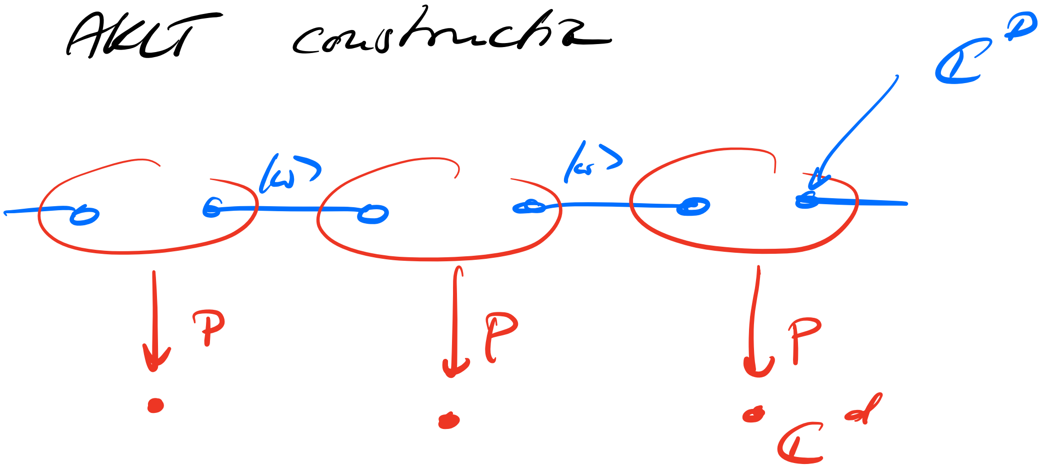
→ points to non-trivial quantum correlations in the system

→ sign of a different type/phase of matter?

2. Solvable MPS models and classification of phases

a) Relation betw. AKLT construction and MPS

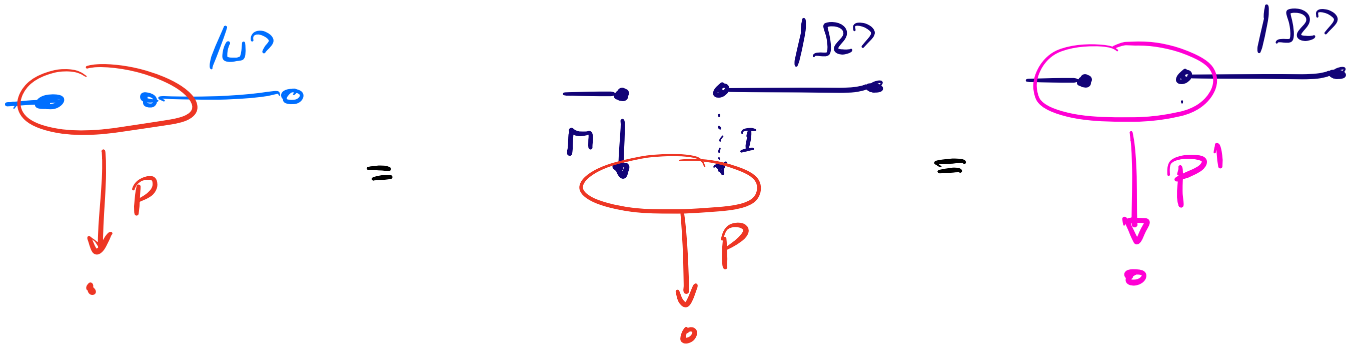
The AKLT construction



with arbitrary P and $|\omega\rangle$ is equivalent to MPS construction.

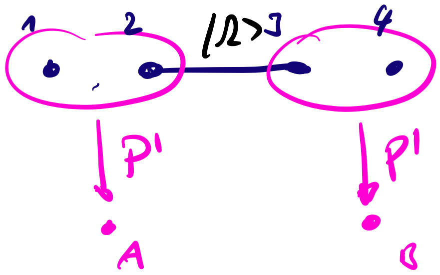
Step I:

Replace $|\omega\rangle$ by $|\Omega\rangle = \frac{1}{\sqrt{D}} |i,i\rangle$ by writing $|\omega\rangle = (\mathbb{1} \otimes \Pi) |\Omega\rangle$ and replacing P by $P' = P(\Pi \otimes \mathbb{1})$.



<u>Step II</u>:

Consider



and write

$$P' = \sum_{i\alpha\beta} A^{i}_{\alpha\beta}\, |i\rangle \langle \alpha,\beta|$$

$$\left(P'_{12\to A} \otimes P'_{34\to B}\right) |\Omega\rangle_{23} =$$

$$= \left(\sum A^{i}_{\alpha\beta} A^{j}_{\gamma\delta}\, |i\rangle_A |j\rangle_B \langle \alpha,\beta|_{12} \langle \gamma,\delta|_{34} \right) \left(\sum_k |k,k\rangle_{23} \right)$$

$$\to \equiv \delta_{\beta\gamma}$$

$$= \sum A^{i}_{\alpha\beta} A^{j}_{\beta\delta}\, |i,j\rangle_{A,B} \langle \alpha,\delta|_{1,4}$$

$$= \sum \left(A^{i} \cdot A^{j}\right)_{\alpha\delta}\, |i,j\rangle_{A,B} \langle \alpha,\delta|_{1,4}$$

$\Rightarrow$ This is the same form as before, with a 2-site map described by $A^{i} \cdot A^{j}$.

We can iterate this to obtain that the state is of MPS form,

$$|\psi\rangle = \sum \mathrm{tr}\left[A^{i_1} A^{i_2} \cdots A^{i_N}\right] |i_1,\dots,i_N\rangle .$$

<u>Note</u>: This also works without tr.

6) Symmetries in MPS

How can we construct MPS with a global on-site symmetry

$$|\psi\rangle = U_g^{\otimes N}|\psi\rangle \quad (\text{or } e^{i\phi_g}|\psi\rangle = U_g^{\otimes N}|\psi\rangle)$$

(with U_g a representation of a sym. group)?

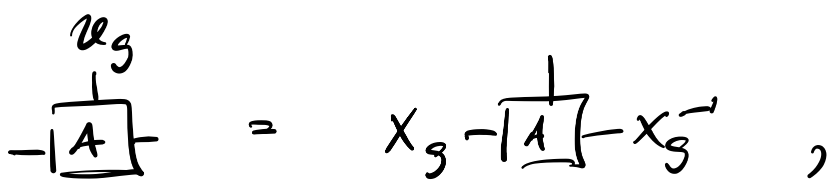
(Similar arguments will also apply to other global symmetries, such as reflection or time reversal = complex conjugation).

This is e.g. important if we want to study unique ground states of symmetric Hamiltonians,

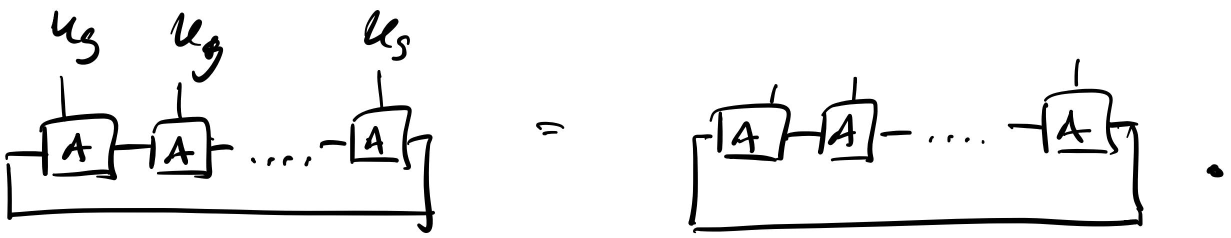
$$[H, U_g^{\otimes N}] = 0\,,$$

as those will also be symmetric.

Observation: If



then

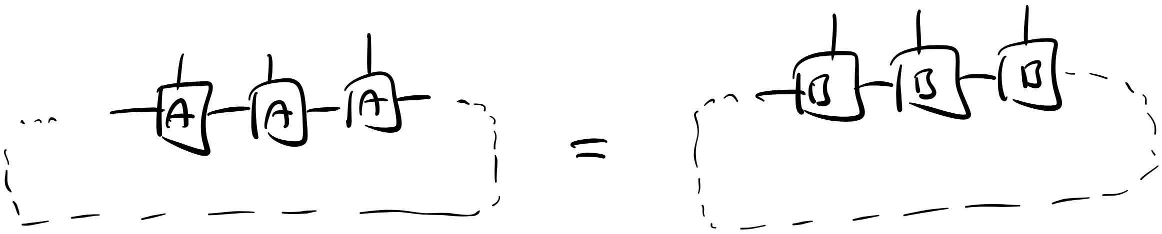


<u>Question</u>: Is this the <u>only way</u> to encode symmetries?

Under suitable (generic) conditions: <u>Yes</u>!

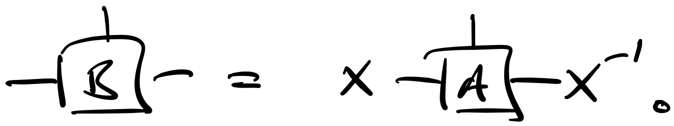
Consequence of "<u>fundamental theorem of MPS</u>": Under suitable (generic) conditions ("injective"/"normal" tensors), the following holds:

If two MPS tensors describe the same state,



for all sizes N,

then $\exists\, X$ invertible:



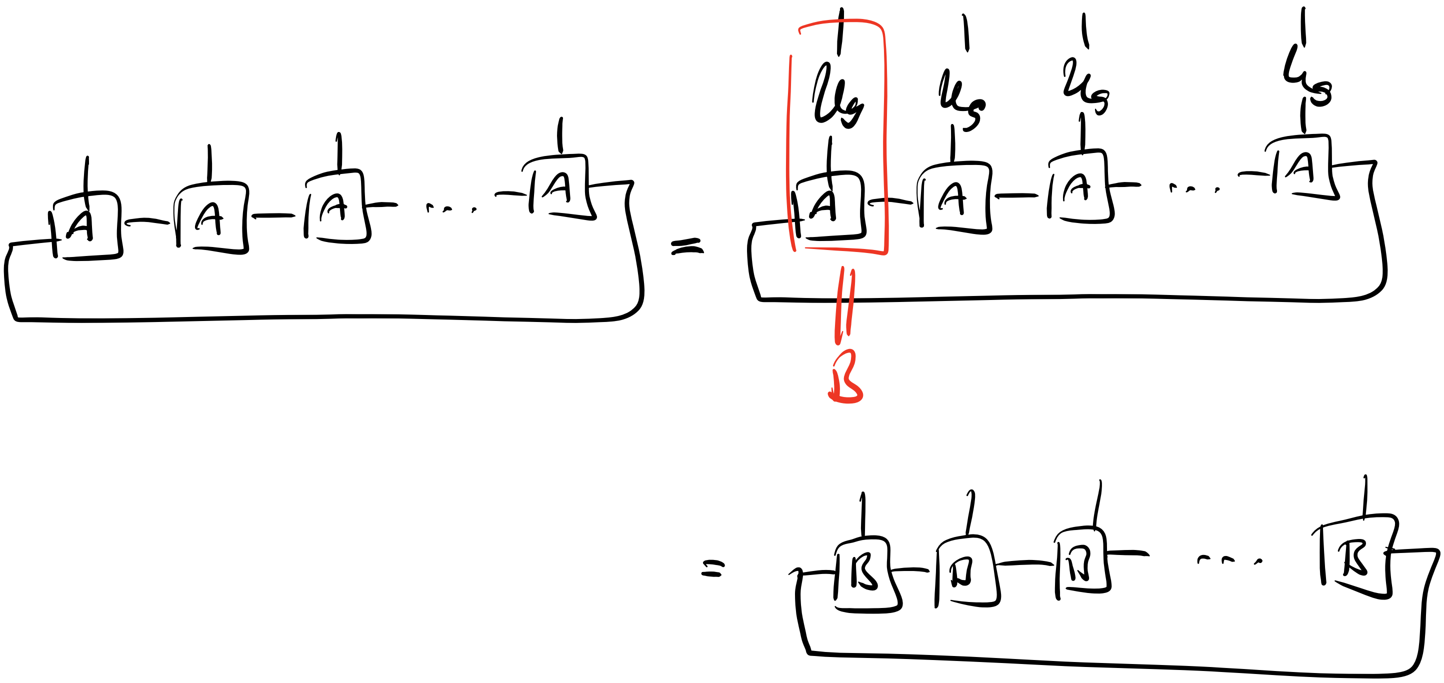
Notes:

- This X is sometimes referred to as a "gauge degree of freedom". Note that e.g. left-/right-canonical form partly fix this gauge (up to a unitary).
- The required conditions hold for a generic MPS.
- A PBC tiv. MPS without long-range order can be always brought into a canonical form (i.e. a new tensor which describes the same state) where it satisfies the required conditions.
- A generalization of the fundamental theorem for general tiv. MPS exists.
- More formal statement/proof → 2nd half of course.

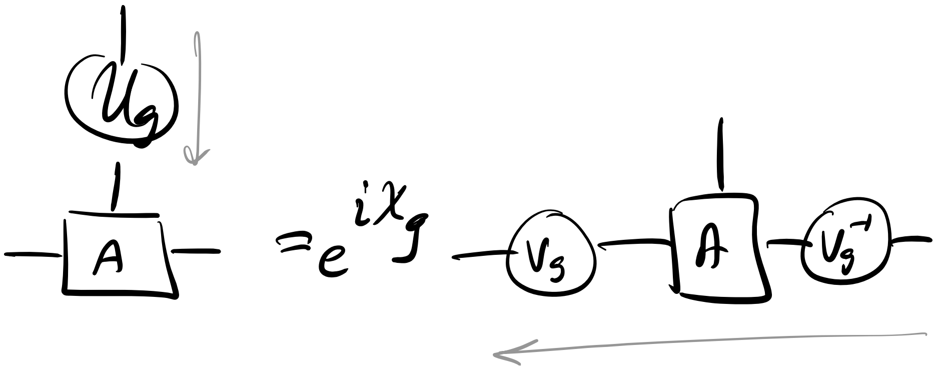
Application to on-site symmetries

(similar for others):

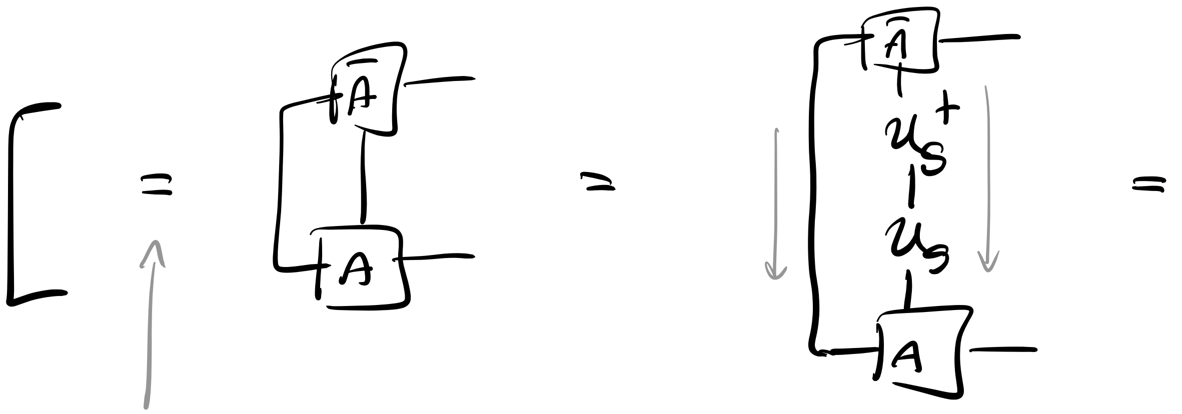
Let $|\psi\rangle = U_g^{\otimes N}|\psi\rangle$, i.e.



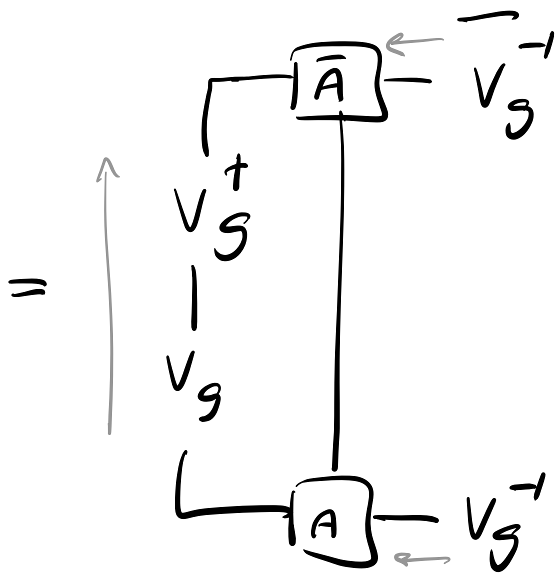
fund. $\Longrightarrow$ thm.

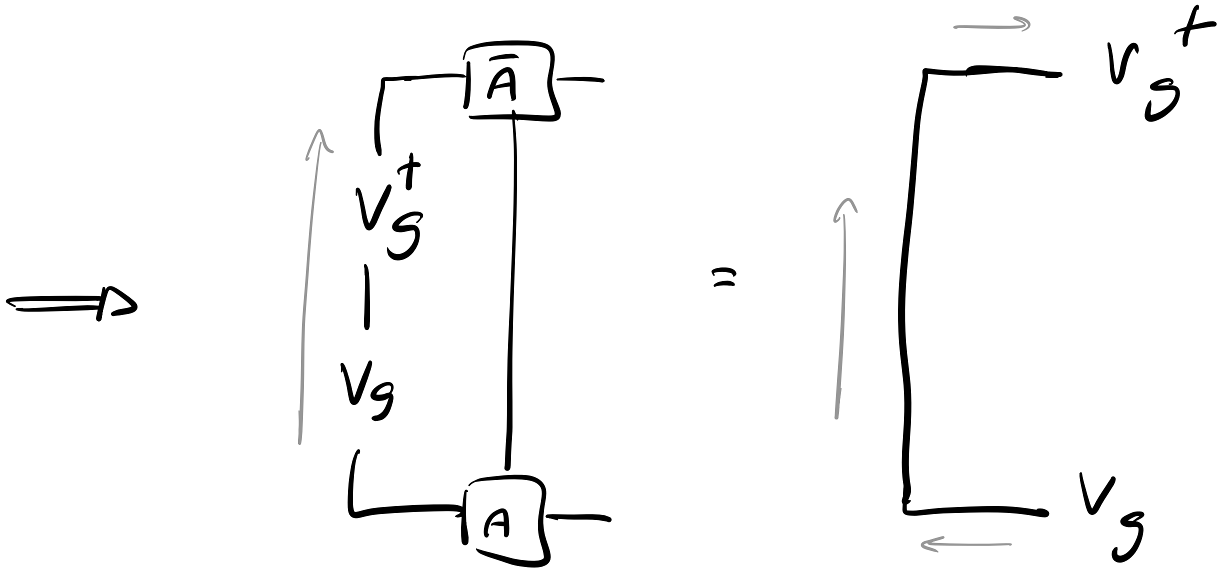


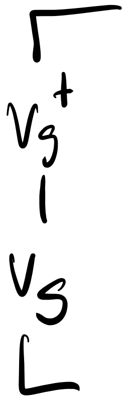
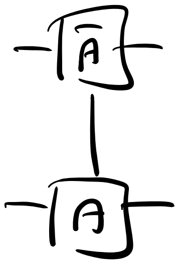
Now let A be in left-can. form:



[leading eigenvector.





$\Longrightarrow$  is eigenvector of 

with eigenvalue 1.

<u>But</u>: Our generic conditions imply that the transfer matrix only has <u>one</u> leading eigenvalue

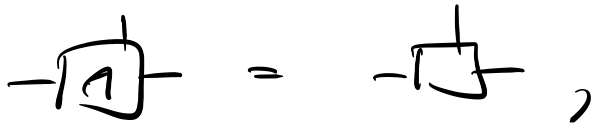
$$\Longrightarrow \quad V_g^{\dagger} V_g = \mathbb{1}$$

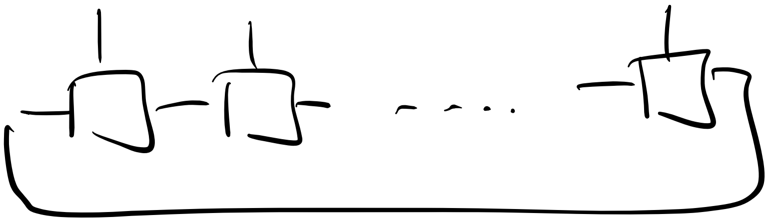
$\Longrightarrow$ <u>V_g is unitary!</u>

c) Parent Hamiltonians and symmetries:

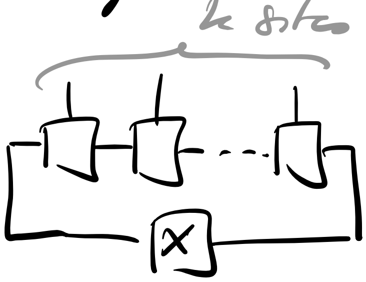
As for the AKLT model, we can construct parent Hamiltonians for any MPS:

Consider a transl. inv. PBC MPS $|\psi\rangle$ w/ tensor

A =  ,

$|\psi\rangle$ = 

Consider k contiguous sites, and define

$S_k :=$ { 

| X $D\times D$ matrix }

Then, $\dim S_k \le D^2$, and $S_k \subset (\mathbb{C}^d)^{\otimes k}$.

$\Rightarrow$ For $d^k > D^2$, S_k is not the full space.

Define $h = \mathbb{1} - \Pi_{S_k}$

proj. onto S_k.

Then, $H = \sum h_i$ has the MPS $|\psi\rangle$

centered at i

as its ground state, $H|\psi\rangle = 0$; $H \geq 0$.

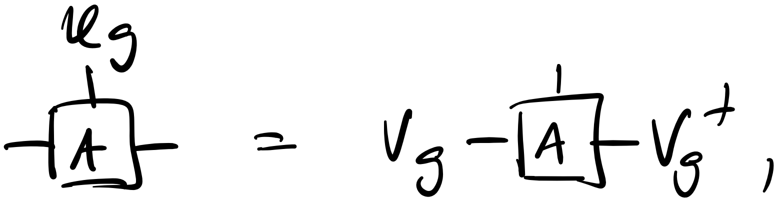
Results (w/out proof): Under the same generic conditions H — with S_k defined for a k which is determined by the properties of the MPS tensor; generically $k = k_0 + 1$, with k_0 the smallest k_0 s.th. $d^{k_0} \geq D^2$ — H has a unique ground state.

Such a Hamiltonian is called a parent Hamiltonian.

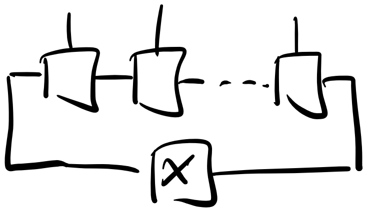
Under the same conditions, H is gapped.

Similar results apply for MPS with long-range order; then, the Hamiltonian has a degenerate ground state & a gap above.

<u>Observation:</u> If $|\psi\rangle$ is symmetric,



, and thus

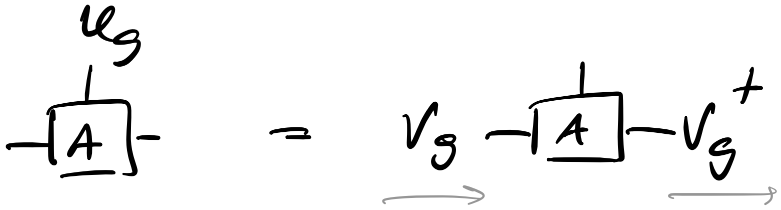
$S_k :=$ $\Big\{$  $\Big|$ X $D\times D$ matrix $\Big\}$

is symmetric (for any $|\phi\rangle \in S_k$, obtained from some X, $U_g^{\otimes k}|\phi\rangle \in S_k$, with boundary $V_g X V_g^\dagger$).

$\Rightarrow$ symmetric MPS have symmetric parent Hamiltonians.

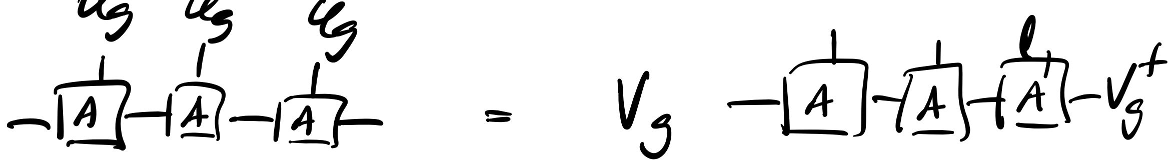
d) Projective representations & the classification of phases

Symmetric MPS (= unique G.S. of gapped symmetric Hamiltonians):

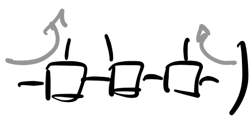


What can we say about V_g?

If we block a few sites



& for generic tensors, this establishes an isomorphism

$u_g^{\otimes k} \cong V_g \otimes \overline{V_g}$ (via the map )

$u_g u_h = u_{gh} \Longrightarrow$

$$(V_g \otimes \overline{V_g})(V_h \otimes \overline{V_h}) = V_{gh} \otimes \overline{V_{gh}}$$

+ unitarity $\Longrightarrow$

$$\boxed{V_g \cdot V_h = e^{i\omega(g,h)} V_{gh}}$$

$\in [0, 2\pi]$

This is called a <u>projective representation</u>.

However, each V_g is only <u>defined</u> up to phase: $V_g \leftrightarrow e^{i\phi_g} V_g$.

In addition, $\omega(g,h)$ must satisfy certain consistency conditions from associativity:

$$V_g (V_h V_k) = (V_g V_h) V_k .$$

The equivalence classes of those $\omega(g,h)$ under $V_g \leftrightarrow e^{i\phi_g} V_g$ characterize inequivalent projective representations.

The diff. equivalence classes are labelled by the 2nd <u>group cohomology</u> $H^2(G, U(1))$.

<u>Classification of phases:</u> For a given symm. group G, MPS + parent Hams. w/ inequivalent proj. reps. V_g^1, V_g^2 correspond to different phases: In order to interpolate along a gapped symmetric path, we need a $V_g^{\text{int.}}$ everywhere, but $V_g^1 \oplus V_g^2$ is a projective rep. if & only if V_g^1 & V_g^2 are in the same equiv. class.

Such phases are called <u>SPT phases</u>.

(SPT = "symmetry-protected topological")

Example: For SO(3), the half-integer and integer spin reps. are in different equiv. classes. In particular, in SO(3), π-rotations about X & Z commute:

$$R_x(\pi) R_z(\pi) R_x(\pi) R_z(\pi) = \mathbb{1},$$

while a possible spin-½ rep. is given by the Pauli matrices σ_x & σ_z, which anticommute:

$$\sigma_x \sigma_z \sigma_x \sigma_z = -\mathbb{1},$$

$\Rightarrow$ The AKLT model is in a different phase ("Haldane phase") than a trivial system.