Entanglement in quantum many-body systems

I. Introduction

1. Quantum many-body systems & lattice models

Quantum many-body (QMB) systems:

many "elementary" particles / constituents interacting.

E.g.: Nuclei + electrons which interact

→ basis of all the "normal" matter around us:

molecules, solids, liquids, gases, ...

Focus of lecture: Solids:

- Electrons + nuclei → form crystal lattice
- Electrons in filled orbitals & in chemical bonds:

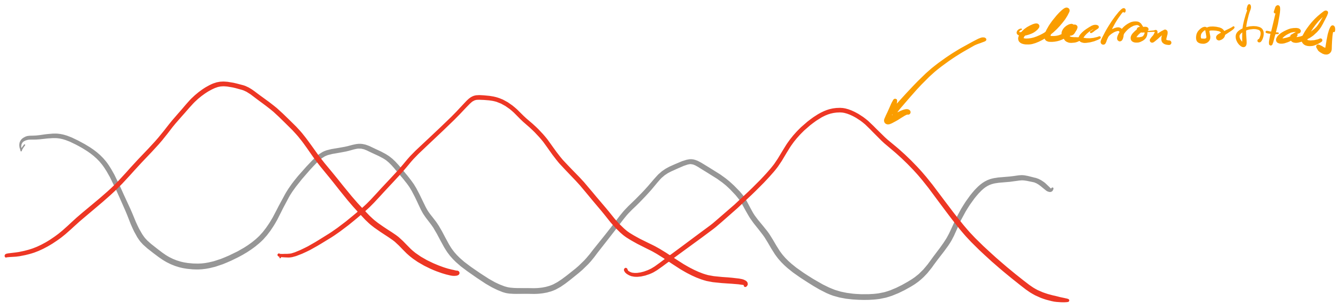
  not relevant at low energies, e.g. room temperature, where crystal stable.

  (Note: typ. binding energy: ~ 1 Hartree = 13.6 eV,

  room temperature: 300K ≈ 25 meV)

- Remaining electrons (in not fully filled orbitals):
  (or holes = missing electrons, e.g. 9d)

  * experience a periodic potential from crystal lattice



large orbitals,
big overlap
(typ. s/p orbitals, since these have large principal quantum number)

↓

lattice of fermionic modes, where fermions can move

↓

often metallic behavior.

— for now not our focus, will return to it later —

small orbitals,
little overlap
(typ. d/f orbitals, which have small principal quantum number → transition group elements)

↓

electrons localized, do not move → fermionic nature irrelevant, but they have a spin-$1/2$.

↓

effective <u>lattice of spins</u>
(typ. insulators).

→ effective model for (part of) such a system:

spin-$1/2$ degrees of freedom on a regular lattice.

(A priori a 3D lattice, but e.g. layered materials can have independent 2D layers; or we can be interested in effects at surfaces / edges: 2D/1D also relevant.)

(<u>Note</u>: There is also other mechanisms which can create such effective spin models – e.g. fermions in a half-filled band; and they can be created in quantum simulators, e.g. in optical lattices.)

2. Quantum Spin Systems

a) Hilbert space of quantum spin systems

What Hilbert space (i.e., wavefunction) do we need to describe a quantum spin system?

A single spin can be in two states,

$$|\uparrow\rangle \quad \text{or} \quad |\downarrow\rangle.$$

We will often use the notation

$$|0\rangle \equiv |\uparrow\rangle \quad \text{and} \quad |1\rangle \equiv |\downarrow\rangle.$$

We can also use a basis notation

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

A general state of one quantum spin is then

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle \in \mathbb{C}^2$$

For two spins, we then have basis states

$$|0\rangle\otimes|0\rangle \equiv |0\rangle|0\rangle \equiv |00\rangle \equiv \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$|0\rangle\otimes|1\rangle \equiv |0\rangle|1\rangle \equiv |01\rangle \equiv \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$|1\rangle\otimes|0\rangle \equiv |1\rangle|0\rangle \equiv |10\rangle \equiv \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

$$|1\rangle\otimes|1\rangle \equiv |1\rangle|1\rangle \equiv |11\rangle \equiv \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

"lexicographic" ordering:
$|00\rangle$
$|01\rangle$
$|10\rangle$
$|11\rangle$

A general state of two qubits is then of the form

$$|\phi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$$

$$\in \mathbb{C}^2\otimes\mathbb{C}^2 = \left(\mathbb{C}^2\right)^{\otimes 2} \cong \mathbb{C}^4$$

(Note: This contains states not of the form $|\phi_1\rangle\otimes|\phi_2\rangle$!)

Basis for N spins:

$$|s_1, s_2, \ldots, s_N\rangle, \quad \text{with} \quad s_i = 0,1 \quad \forall i:$$

$|00 \ldots 00\rangle$
$|00 \ldots 01\rangle$
$|00 \ldots 10\rangle$
$\vdots$
$|11 \ldots 11\rangle$

2^N orthogonal basis vectors

(Note: If written as vector, order components as here.)

Most general state

$$|\phi\rangle = \sum_{s_i=0,1} c_{s_1,\dots,s_N} |s_1, s_2, \dots, s_N\rangle$$

$$\in \underbrace{\mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2}_{N \text{ times}} = (\mathbb{C}^2)^{\otimes N} \cong \mathbb{C}^{(2^N)}$$

2^N - dimensional vector!

State of a spin system with N spins lives in an exponentially big Hilbert space of dimension 2^N!

More generally, if we have a d-level system, $d \geq 2$, at each lattice site (e.g. optical lattices, effective degrees of freedom), with basis $|0\rangle, \dots, |d-1\rangle$, the state is

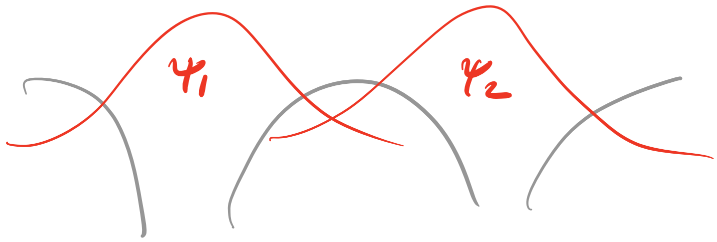
$$|\phi\rangle = \sum_{s_i=0}^{d-1} c_{s_1\dots s_N} |s_1, \dots, s_N\rangle \in (\mathbb{C}^d)^{\otimes N} \cong \mathbb{C}^{(d^N)}$$

i.e., it lives in a d^N - dim. Hilbert space.

b) Interactions

To study the physics of a quant. system, we need to know its Hamiltonian – here, how the spins interact.

First, consider two spins:



also used in optical lattices!

One possible mechanism (not most common, but easiest to explain): Direct exchange.

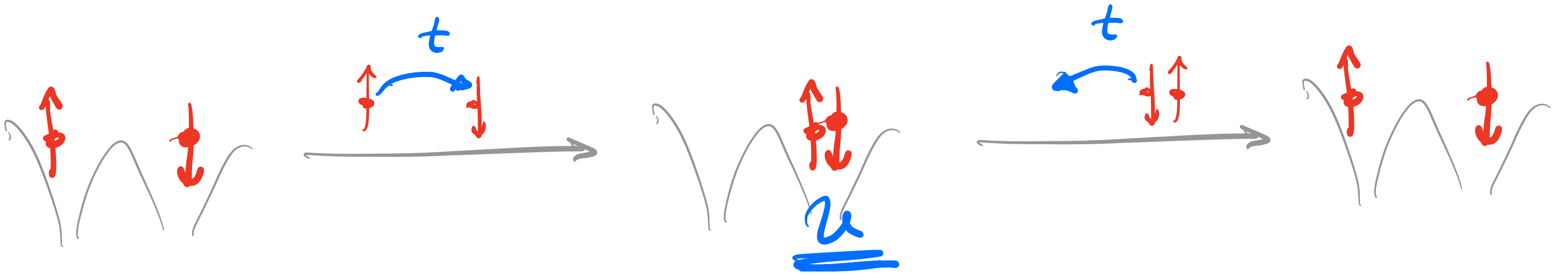
- orbitals ψ_1 and ψ_2 overlap
  $\Rightarrow$ possibility for electron to tunnel from $1 \leftrightarrow 2$ with tunneling rate t.

- Consider a process where electron 1 tunnels to 2
- Can only happen if the two electrons form a singlet (Pauli exclusion principle),

$$|\phi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

- If both electrons are at the same site, they experience strong Coulomb repulsion U.

- $U \gg t$: ground space has exactly one electron per site, but there is an energy correction from 2nd order perturbation theory:



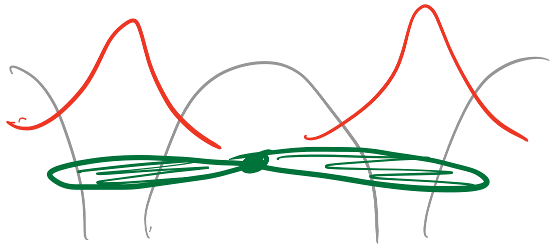
correction from 2nd order perturbation theory:

$$\Delta E = -\frac{t^2}{U}$$

- We thus find: energy of singlet state $|\phi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ lower by $-\frac{t^2}{U}$

$\Rightarrow$ antiferromagnetic Heisenberg interaction.

The same, or similar, interactions (including ferromagnetic ones) can be obtained from a range of other mechanisms, e.g.



- intermediate orbitals which induce an effective coupling
- coupling through intermediate coupling to a band of electrons — the RKKY interaction (Ruderman - Kittel - Kasuya - Yosida)

Further reading: W. Nolting, A. Ramakanth: Quantum Theory of Magnetism (Springer 2009)

What is the general structure of interactions in a quantum spin system?

→ Locality: interactions only couples nearby spin (or strength decays rapidly with distance)

→ few-body: interactions only couple a small number (typ. 2) spins.

→ <u>symmetry</u>: interactions generally have the symmetries of the setup (lattice, ...)

How does a general 2-body interaction look like?

$$h: \mathbb{C}^2 \otimes \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2$$

↑ Hamiltonian: 4×4 - matrix

We can express h using spin operators:

$$S^x = \frac{1}{2}\sigma^x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad S^y = \frac{1}{2}\sigma^y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix};$$

$$S^z = \frac{1}{2}\sigma^z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

and $\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The spin operator S^α acting on site 2 is given by

$$S_2^\alpha = \mathbb{1} \otimes S^\alpha$$

$$\equiv \mathbb{1}_1 \otimes S_2^\alpha = \left(\begin{array}{c|c} 1\cdot S_2^\alpha & 0\cdot S_2^\alpha \\ \hline 0\cdot S_2^\alpha & 1\cdot S_2^\alpha \end{array}\right) \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array}$$

shorthand notation

$$= \left(\begin{array}{c|c} S_2^\alpha & \\ \hline & S_2^\alpha \end{array}\right)$$

and similarly:

$$S_1^\alpha = S^\alpha \otimes \mathbb{1} \equiv S_1^\alpha \otimes \mathbb{1}_2 ,$$

$$\text{e.g.} \quad S_1^x = \frac{1}{2}\left(\begin{array}{c|c} 0\cdot\mathbb{1} & 1\cdot\mathbb{1} \\ \hline 1\cdot\mathbb{1} & 0\cdot\mathbb{1} \end{array}\right) = \frac{1}{2}\left(\begin{array}{c|c} 0 & \begin{smallmatrix}1&0\\0&1\end{smallmatrix} \\ \hline \begin{smallmatrix}1&0\\0&1\end{smallmatrix} & 0 \end{array}\right).$$

We can also act with S^α on 1 & S^β on 2:

$$S_1^\alpha \cdot S_2^\beta = S^\alpha \otimes S^\beta$$

$$\equiv S_1^\alpha \otimes S_2^\beta = \frac{1}{2}\left(\begin{array}{c|c} 0 & S_2^\beta \\ \hline S_2^\beta & 0 \end{array}\right).$$

shorthand notation w/out $\otimes$

e.g. $S_1^\alpha = S_1^x$

Examples:

$$S_1^x \cdot S_2^x = \frac{1}{4}\begin{pmatrix} & 0 & & 1 \\ & 0 & 1 & \\ & 1 & 0 & \\ 1 & & & 0 \end{pmatrix};$$

$$S_1^y \cdot S_2^y = \frac{1}{4}\begin{pmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{pmatrix};$$

$$S_1^z \cdot S_2^z = \frac{1}{4}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$$

Note: The spin operator in a general direction $\vec{r} = (r_x, r_y, r_z)$, $\|\vec{r}\| = 1$, is

$$r_x \cdot S^x + r_y S^y + r_z S^z = \vec{r} \cdot \vec{S},$$

with $\vec{S} = (S^x, S^y, S^z)$,

Rotations O in real space transform $\vec{r} \cdot \vec{S}$ by rotating $\vec{r}$ – and thus equivalently $\vec{S}$:

$$\vec{r} \cdot \vec{S} \xmapsto{\;O\;} (O\vec{r}) \cdot \vec{S} = \vec{r} \cdot (O^T \vec{S})$$

What are some prototypical simple interactions?

- The derivation before — $|\phi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ gets energy $-t^2/u$ — is:

$$h = \underbrace{-\frac{t^2}{u}|\phi\rangle\langle\phi|}_{\text{energy } -\frac{t^2}{u} \text{ to } |\phi\rangle,\ 0 \text{ to the other states}} = -\frac{t^2}{u}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\frac{(0\ 1\ -1\ 0)}{\sqrt{2}}$$

$$= -\frac{t^2}{2u}\begin{pmatrix}0&&&\\&1&-1&\\&-1&1&\\&&&0\end{pmatrix} = \frac{t^2}{2u}\begin{pmatrix}0&&&\\&-1&1&\\&1&-1&\\&&&0\end{pmatrix}$$

- The only fully rotationally invariant interaction:

$$h = J\cdot\left(S_1^x S_2^x + S_1^y S_2^y + S_1^z S_2^z\right)$$

$$\equiv J\cdot\left(\vec{S}_1 \cdot \vec{S}_2\right)$$

$$= \frac{J}{4}\left[\begin{pmatrix} 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0\\ 1&0&0&0 \end{pmatrix} + \begin{pmatrix} 0&0&0&-1\\ 0&0&1&0\\ 0&1&0&0\\ -1&0&0&0 \end{pmatrix} + \begin{pmatrix} 1&0&0&0\\ 0&-1&0&0\\ 0&0&-1&0\\ 0&0&0&1 \end{pmatrix}\right]$$

$$= \frac{J}{4}\begin{pmatrix} 1&0&0&0\\ 0&-1&2&0\\ 0&2&-1&0\\ 0&0&0&1 \end{pmatrix}$$

$$= \frac{J}{4}\left[\begin{pmatrix} 0&0&0&0\\ 0&-2&2&0\\ 0&2&-2&0\\ 0&0&0&0 \end{pmatrix} + \underbrace{\begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}}_{\text{constant!}}\right]$$

$$= \frac{J}{2}\begin{pmatrix} 0&0&0&0\\ 0&-1&1&0\\ 0&1&-1&0\\ 0&0&0&0 \end{pmatrix} + \frac{J}{4}$$

$\Rightarrow$ <u>same</u> as $-\frac{t^2}{u}|\phi\rangle\langle\phi|$ (up to constant)

Heisenberg interaction

$J>0$: antiferromagnetic

$J<0$: ferromagnetic

Eigenvalues of operator

$$\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{4}\begin{pmatrix} 1 & & & \\ & -1 & 2 & \\ & 2 & -1 & \\ & & & 1 \end{pmatrix}:$$

$1 \times \left(-\frac{3}{4}\right)$, with eigenvector $\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = |01\rangle - |10\rangle$) total spin = 0

$3 \times \left(+\frac{1}{4}\right)$, with eigenvectors (total spin = 1)

$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |00\rangle$ — $S_z = +1$

$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |11\rangle$ — $S_z = -1$

$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = |01\rangle + |10\rangle$ — $S_z = 0$

Important: To preserve rotational symmetry, $\frac{|01\rangle + |10\rangle}{\sqrt{2}}$ must go with the ferromagnetic states: classical intuition misleading!

$\Rightarrow$ form of quantum correlations (entanglement) plays an essential role!

Other important interactions:

Ising interaction: $h = S_1^x \cdot S_2^x$

(or $S_1^z \cdot S_2^z$, ...)

XX interaction: $h = S_1^x \cdot S_2^x + S_1^y \cdot S_2^y$

XXZ interaction: $h = S_1^x \cdot S_2^x + S_1^y \cdot S_2^y + \Delta S_1^z \cdot S_2^z$

... those have a preferred axis/plane.

How do these act on the full N-spin Hilbert space?

$$|\phi\rangle = \sum c_{s_1 \dots s_N} |s_1, s_2, \dots, s_N\rangle$$

$h_{12} |\phi\rangle$ acts only on s_1, s_2, and leaves other s_i invariant:

$$h_{12} |\phi\rangle = \sum c_{s_1 \dots s_N} \left(h | s_1, s_2 \rangle \right) \otimes | s_3, s_4 \dots \rangle$$

$$= \sum c_{s_1 \dots s_N} \left(h_{12} \right)_{s_1 s_2}^{s_1' s_2'} | s_1', s_2', s_3, \dots \rangle$$

That is: h_{12} should be understood as

$$h_{12} \otimes \mathbb{1}_3 \otimes \mathbb{1}_4 \otimes \dots \otimes \mathbb{1}_N .$$

Or we can do this right at the level of spin operators,

$$S_i^\alpha = \mathbb{1}_1 \otimes \dots \otimes \mathbb{1}_{i-1} \otimes S_i^\alpha \otimes \mathbb{1}_{i+1} \otimes \dots ,$$

and define h_{ij} using these S_i^α, e.g.,

$$h_{ij} = J \vec{S}_i \cdot \vec{S}_j .$$

Can couple arbitrary spins this way, but typ. Hamiltonian should be local (→ mechanism behind nat.)

Total Hamiltonian: Sum of all (local) terms,

$$H = \sum_k h_k \overset{\text{e.g.}}{=} \sum J_{ij} \vec{S}_i \cdot \vec{S}_j$$

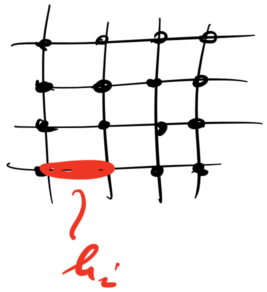
sum over all local terms → k

J_{ij} ← should decay with distance.

c) Study of quantum spin systems

Spin system:

$$\mathcal{H} = (\mathbb{C}^d)^{\otimes N} \; ;$$ lattice geometry



$$H = \sum_i h_i$$ local / quasi-local interactions

H is typically transl. invariant, i.e. $h_i \equiv h$, centered at position i — e.g. Heisenberg coupling, ...

time-indep. Schrödinger equation

$$H|\psi\rangle = E|\psi\rangle; \qquad |\psi\rangle \in \mathcal{H}.$$

in particular: lowest eigenvalue E_0 and corresponding eigenvector $|\psi_0\rangle$:

ground state $|\psi_0\rangle$, ground state energy E_0

– describes system at sufficiently low temperatures.

Thermal state

$$\rho = \frac{e^{-\beta H}}{Z} \; ; \quad Z = \mathrm{tr}\left(e^{-\beta H}\right)$$

$$\beta = \frac{1}{kT}$$

significantly more complex than $|\psi_0\rangle$:
$2^N \times 2^N$ - matrix.
For T small enough: $\rho \approx |\psi_0\rangle\langle\psi_0|$.

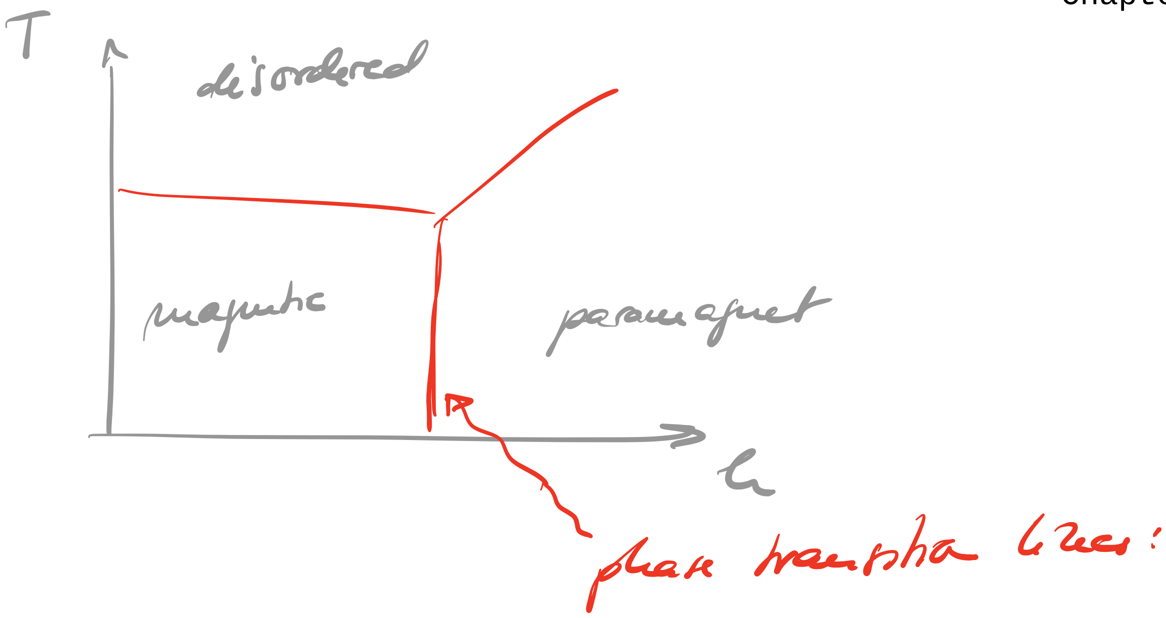
Key questions to ask about system
(e.g. for ground or thermal state):

What type of order (phase) does system exhibit?

- long-range magnetic order
- no magnetic order
- other types of order?!

... as a function of T, or of some parameter in H, such as different couplings, a magnetic field

$$H' = H - h \cdot \sum_i S_i^z \; , \quad \text{or} \quad H' = H - \sum_i \vec{h} \cdot \vec{S}_i \; , \ldots$$



Where are the phase transitions?

What properties do they have?

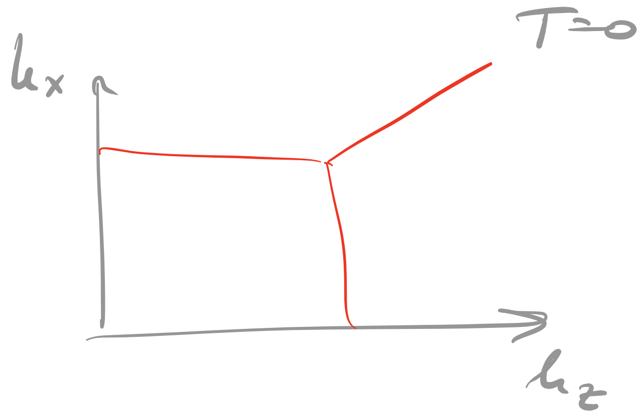
Focus: Quantum Matter – materials where quantum effects play an <u>essential</u> role.

⇒ this is more prominent at low T

($kT \ll$ energy scales of H (cf. later))

(Why? → cf. later: at large T, quantum correlations – entanglement – vanish.)

⇒ Special interest in physics at $T=0$, i.e. ground state properties & phase diagram.



"quantum phases"

"quantum phase transitions"

(Important point: Are properties at T=0 stable against small T>0? → later!)

What properties are we interested in?

- magnetic order:

e.g. average magnetization

$$\vec{m} = \frac{1}{N}\sum_i \langle \vec{S}_i \rangle = \begin{cases} = 0 \\ \neq 0 \end{cases} ?$$

↑ ferromagnetic

or, more general,

$$\vec{m}(k) = \frac{1}{N}\sum_j e^{ikj}\langle \vec{S}_j \rangle = ?$$

e.g. for 2D, $k=(\pi,\pi)$: "staggered magnetization", detects antiferromagnetic order.

- correlations between spins

* $\langle S_i^\alpha \cdot S_j^\beta \rangle$ — for transl. invariant systems, this only depends on $i-j$, or (if we also have reflection sym.) on $|i-j|$.

* average $\frac{1}{N} \sum_{ij} \langle S_i^\alpha S_j^\beta \rangle$ (= O(1) if correlations decay exponentially, O(N) with long-r. order)

* "structure factor" $S(k) = e^{ik(i-j)} \langle S_i^\alpha S_j^\beta \rangle$

→ encodes information about magnetic order

→ $S(k)$ can be measured with elastic neutron scattering

→ behavior of correlations, e.g.

$$\langle S_i^\alpha S_j^\alpha \rangle \sim e^{-|i-j|/\xi}$$

gives correlation length ξ, which diverges at phase trans. & gives extra info. about type of transition.

https://www.ericmfischer.com/project/exact-sampling-cluster-sampling/

- ground state energy E_0

By itself meaningless, but derivatives with respect to parameters (fields, ...) encode information

(cf. free energy $F = -kT \ln Z$)
$Z = e^{-\beta H}$

e.g.: $H' = H + \lambda V$, e.g. $V = \sum S_i^z$:

$$\left.\frac{dE_0(H')}{d\lambda}\right|_{\lambda=0} = \left.\frac{d}{d\lambda}\right|_{\lambda=0} \left(\langle \psi_0(\lambda) | H + \lambda V | \psi_0(\lambda) \rangle \right)$$

$$= \langle \psi_0(0) | V | \psi_0(0) \rangle$$

(other terms vanish as $\frac{d|\psi_0\rangle}{d\lambda}$ must be orthogonal to $|\psi_0\rangle$ due to normalization)

- Finally, we might also be interested in other questions:
  - time evolution, e.g. after change of H ("quench"), or flipping a spin
    can be meas. w/ in-elastic neutron scattering

- excited states:

$$H|\psi_{k,E}\rangle = E_k |\psi_{k,E}\rangle$$

with momentum $T|\psi_{k,E}\rangle = e^{ik}|\psi_{k,E}\rangle$

(T: translation operator)

- effects of disorder in H
- properties of thermal states
- ... and much more!

For the beginning, key questions will be:

- what is the ground state
- what are its properties

This will also form the basis for many of the other questions.

d) The spectral gap

What characterizes a phase transition?

- Divergence of correlation length
- discontinuity of derivatives of certain quantities.
- ...

Phase transition: Small change in parameters can give rise to large (sudden) change in physical properties – the system is unstable.

Inside a phase: System should only react weakly to small perturbations, i.e. the properties and thus the system are stable against perturbations.

How can we characterize (in-)stability to small perturbations $H \longrightarrow H' = H + \varepsilon V$ in a simple way?

Perturbation theory:

H: ground state $|\psi\rangle$ w/ energy E_0,
ex. states $|\phi_i\rangle$ w/ energy E_i (sorted: $E_i \le E_{i+1}$)

H': ground state $|\psi'\rangle$

$$|\psi'\rangle = \underbrace{-\varepsilon \sum_i \frac{|\phi_i\rangle\langle\phi_i|V|\psi\rangle}{E_i - E_0}}_{\text{change in state!}} + |\psi\rangle + \dots$$

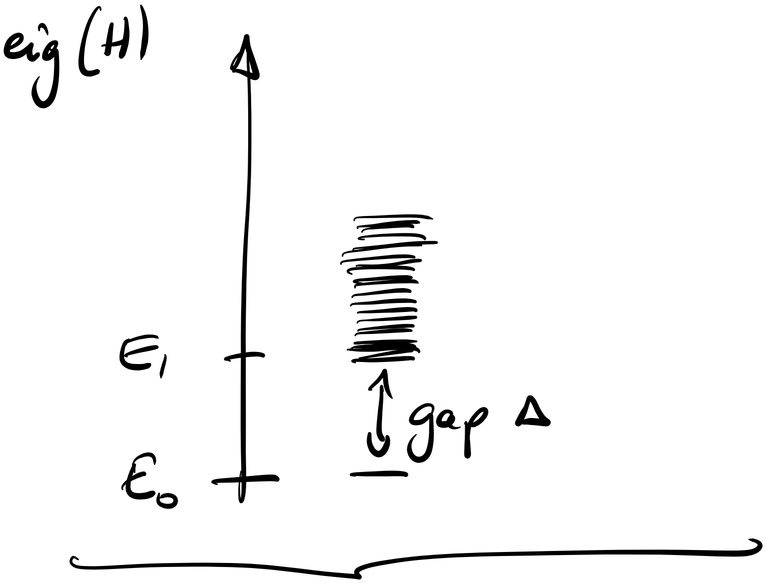
$$\| |\psi'\rangle - |\psi\rangle \| \le \varepsilon \cdot \sum_i \frac{\| |\phi_i\rangle\langle\phi_i| V |\psi\rangle \|}{\underbrace{E_i - E_0}_{\ge E_1 - E_0 =: \Delta}} + O(\varepsilon^2)$$

$$\le \frac{\varepsilon}{\Delta} \cdot \text{const.}$$ ← depends on V!

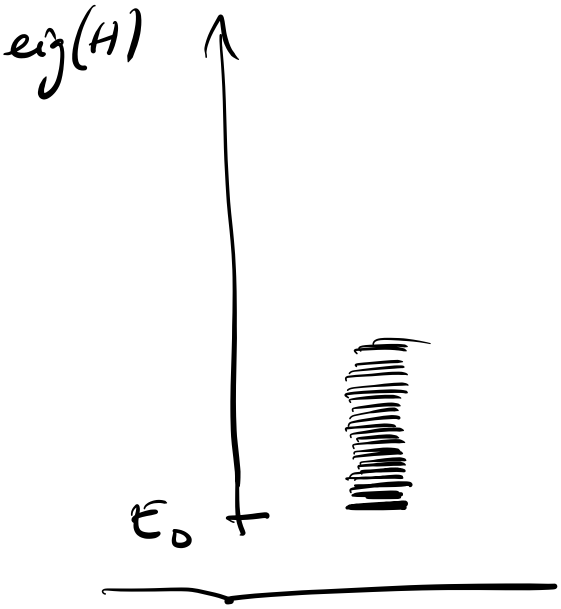
(& higher orders scale with $\left(\frac{\varepsilon}{\Delta}\right)^k$!)

$\Longrightarrow$ If the "energy gap" (or: "spectral gap", or "gap") of H is sufficiently large, then

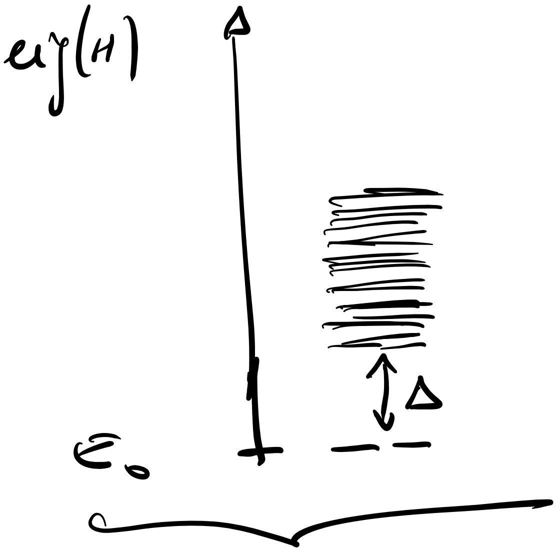
$$\frac{\varepsilon}{\Delta} \ll 1 \quad \text{for} \quad \varepsilon \ll 1 .$$



gapped Hamiltonian



gapless Hamiltonian



gapped with degenerate ground state

Definition: We call a Hamiltonian (~~fam~~, a family of Hamiltonians)

$$H = \sum_{i=1}^{N} h_i$$

on a lattice of size N **gapped** if the gap $E_1(N) - E_0(N) = \Delta(N)$ on a lattice of size N is lower bounded:

$$\Delta(N) \geq \Delta > 0$$

$$(\text{typ., } \Delta(N) \longrightarrow \Delta).$$

We call Δ the **gap (or energy gap, spectral gap)** of H.

This can also be extended to systems with k degenerate (or almost degenerate, as $N \to \infty$) ground states; then, $\Delta(N) = E_{k+1}(N) - E_k(N)$.

Gapless (or critical) systems are those where

$$\Delta(N) \longrightarrow 0 \quad \left(\text{often, } \Delta(N) \sim \frac{1}{\text{poly}(N)}\right).$$

We can define (gapped) quantum phases as regions in parameter space where H is gapped, and the boundaries (transitions) between them as the lines where H is gapless.

Intuition – cf. above: A gap ensures stability of the phase, as the prefactor $\left(\frac{\varepsilon}{\Delta}\right)^k$ in the perturbation series vanishes.

But this is not rigorous, since typ. V is extensive (e.g.: $H' = H + \varepsilon \underbrace{\sum \sigma_i^z}_{\equiv V}$),

and thus $\| V|\psi\rangle \| \propto N$. Thus, higher order terms can in fact get larger (as the bounds scale as $\left(\frac{\varepsilon}{\Delta}\right)^k N^k$!)

Should still be true if the terms in V don't "conspire".

__Proofs__ of such stability possible in certain cases under some additional reasonable assumptions:
Then, one can show that

$$H'(\varepsilon) = H + \varepsilon \sum_i V_i$$

is still gapped for small enough ε, and then, the ground states of H and $H'(\varepsilon)$ only differ inside a "light cone" whose size depends on ε and Δ (up to small corrections):
Thus, the properties of the ground state do not change abruptly, and in particular, no long-range correlation can appear (or disappear).

(Further reading:

https://arxiv.org/abs/1001.0344

for the stability of the gap, and

https://arxiv.org/abs/cond-mat/0503554

for the consequence that the state only changes inside a "light cone".)

A further consequence of a gap is that for low enough temperature,

$$\rho(T) = \frac{e^{-\beta H}}{\mathrm{tr}(e^{-\beta H})} \approx |\psi_0\rangle\langle\psi_0|,$$

with $|\psi_0\rangle$ the ground state (this requires one extra reasonable assumption on the density of states) — i.e., we don't need to cool to $T=0$ to be (effectively) in the ground state.

e) Summary: Setup & Question

- Quantum spin system $\mathcal{H} = (\mathbb{C}^d)^{\otimes N}$.
- Local Hamiltonian $H = \sum h_i$.
- Determine properties of ground state & spectral properties of H.

Q: How can we deal with the exp. dimension d^N of the underlying Hilbert space $\mathcal{H}$?

Observation: $H = \sum_{i=1}^{N} h_i$ specified by $O(N)$ parameters & we care about ground state ⟹ only a small fraction of states in $\mathcal{H}$ actually relevant!

What singles out the relevant states?

⟹ The structure of their q. correlations — **entanglement!**