# 4. POVM measurements

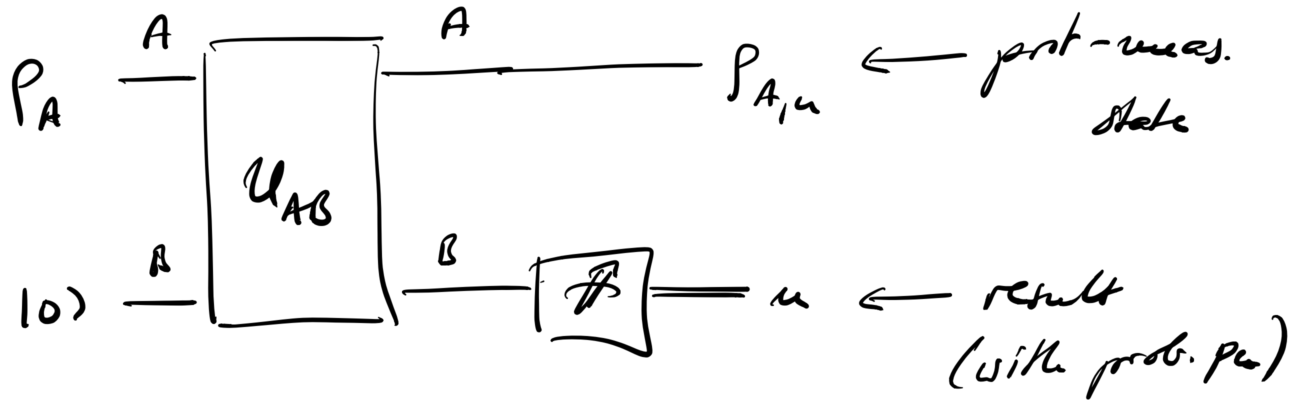
Seen previously: Adding a 2nd system B gives more rich situation.

Thus natural question: What measurements can we do by adding an extra system?

Idea: i) Add auxiliary system ("ancilla") B in state $|0\rangle$

ii) Act w/ unitary $U_{AB}$ on system + ancilla

iii) measure B in $\{|0\rangle, |1\rangle, \ldots, |d_B - 1\rangle\}$




Analyze scheme:

Post-meas. state (unnormalized) is:

$$\tilde{\rho}_n^A = \langle n|_B \, U \, (\rho_A \otimes |0\rangle\langle 0|_B) \, U^\dagger |n\rangle_B$$

$$= \langle \mu|_B U|0\rangle_B \; \rho_A \; \langle 0|_B U^\dagger|\mu\rangle_B$$

$$= \Pi_\mu \rho_A \Pi_\mu^\dagger ,$$

where we have defined

$$\Pi_\mu := \langle \mu|_B U|0\rangle_B \equiv (I_A \otimes \langle \mu|_B)\, U\, (I_A \otimes |0\rangle_B)$$

Then, $p_\mu = \operatorname{tr} \tilde{\rho}_\mu^A = \operatorname{tr}(\Pi_\mu \rho_A \Pi_\mu^\dagger) = \operatorname{tr}(\Pi_\mu^\dagger \Pi_\mu \rho_A)$,

is the probability for outcome $\mu$,

and $\rho_\mu^A = \frac{1}{p_\mu} \tilde{\rho}_\mu^A$ the post-measurement state.

It holds that

$$\sum \Pi_\mu^\dagger \Pi_\mu = \sum_\mu \langle 0|_B U^\dagger|\mu\rangle_B \langle \mu|_B U|0\rangle_B$$

$$= \langle 0|_B U^\dagger U|0\rangle_B$$

$$= I_A ,$$

and further $\Pi_\mu^\dagger \Pi_\mu \geq 0$.

(Note: The former implies

$\sum p_\mu = \sum \operatorname{tr}(\Pi_\mu^\dagger \Pi_\mu \rho) = \operatorname{tr}(\sum \Pi_\mu^\dagger \Pi_\mu \rho) = \operatorname{tr}(\rho) = 1$ ).

**Definition:** A set $\{F_m\}$ of operators, $F_m \geq 0$, $\sum F_m = I$, is called a positive operator-valued measure (POVM).

**Note:** $F_m := M_m^\dagger M_m$ forms a POVM. If we only care about the post-meas. prob. $p_m = tr(F_m \rho)$, then the measurement is fully characterized by the POVM $\{F_m\}$.

**Definition:** A POVM measurement is given by a set of operators $\{M_m\}$ with $\sum M_m^\dagger M_m = I$, with outcome probabilities $p_m = tr(M_m^\dagger M_m \rho)$ and post-measurement states $\rho_m = \frac{1}{p_m} M_m \rho M_m^\dagger$.

**Alternative Definition:** A POVM measurement is given by a set of operators $\{F_m\}$, $F_m \geq 0$, $\sum F_m = I$, with outcome probabilities $p_m = tr(F_m \rho)$.

Relation of the two definitions, & with the initial unitary + ancilla construction:

<u>i)</u> Can any $F_u \geq 0$ be written as $F_u = \Pi_u^\dagger \Pi_u$ ?

Yes – e.g., take $\Pi_u = \sqrt{F_u}$ .

(Unique up to isometric degree of freedom, since

$$\Pi_u = \mathcal{U}_u \sqrt{\Pi_u^\dagger \Pi_u} \quad \text{(the \underline{polar decomposition})}.$$

<u>ii)</u> Can any POVM meas. $\{\Pi_u\}_{u=0}^{N-1}$, $\sum_{u=0}^{N-1} \Pi_u^\dagger \Pi_u = I$, be realized via ancilla + unitary?

$$X := \begin{pmatrix} \Pi_0 \\ \Pi_1 \\ \vdots \\ \Pi_{N-1} \end{pmatrix} \qquad \sum \Pi_u^\dagger \Pi_u = I \iff X \text{ has orthogonal columns}$$

$\Longrightarrow$ X can be extended to a unitary $\mathcal{U}$ by adding further columns,

$$\mathcal{U} = \begin{array}{c|c} & |0\rangle_B \quad |1\rangle_B \quad \cdots \\ \langle 0|_B \\ \langle 1|_B \\ \vdots \\ \langle d-1|_B \end{array} \begin{pmatrix} \Pi_0 & \cdots \\ \Pi_1 & \cdots \\ \vdots & \\ \Pi_{N-1} & \cdots \end{pmatrix}$$

... this can be understood as a unitary acting on system + ancilla $B$ with dim. $d_B = N$.

$$\Longrightarrow \quad \langle n|_B \, \mathcal{U} \, |0\rangle_B = \Pi_n .$$

$\Longrightarrow$ Any POVM meas. $\{\Pi_n\}$ can be realized by adding ancilla, doing a unitary $\mathcal{U}$ on system + ancilla, and projectively measuring ancilla in $\{|0\rangle, \dots, |d_B - 1\rangle\}$ basis.

This is also known as Naimark's theorem.

Note: The "old-style" measurements where the $\Pi_n = E_n$ (or equivalently $F_n = E_n$) are also called projective measurements.

Is this the most general type of measurement?

i) Minimal axioms for q.m. measurements:

Measurements are linear functionals

$$\rho \longmapsto p_u(\rho),$$

which map states to outcome probabilities, such that

$$p_u(\rho) \geq 0 \qquad \forall \rho \geq 0,\ tr(\rho)=1$$

and

$$\sum p_u(\rho) = 1 \qquad \forall \rho \geq 0,\ tr(\rho)=1.$$

ii) Linear functionals $\rho \mapsto p_u(\rho)$ are of the form $p_u(\rho) = tr(F_u \rho)$.

(E.g. by using a basis where $\rho = |e_i\rangle\langle e_j|$)

unit vectors.

iii) We can w.l.o.g. choose

$$F_u = F_u^{+}.$$

Otherwise, write

$$F_k = \underbrace{\tfrac{1}{2}(F_k + F_k^\dagger)}_{\text{herm. part}} + \underbrace{\tfrac{1}{2}(F_k - F_k^\dagger)}_{\text{anti-herm. part}},$$

and

$$\operatorname{tr}\left((F_k - F_k^\dagger)\rho\right) = \operatorname{tr}(F_k\rho) - \operatorname{tr}(F_k^\dagger\rho)$$

$$= \operatorname{tr}(F_k\rho) - \underbrace{\overline{\operatorname{tr}(F_k\rho^\dagger)}}_{\rho=\rho^\dagger, \text{ and } \operatorname{tr}(F_k\rho)\geq 0}$$

$$= \operatorname{tr}(F_k\rho) - \operatorname{tr}(F_k\rho) = 0,$$

$\Rightarrow \operatorname{tr}(F_k\rho) = \operatorname{tr}\left(\tfrac{1}{2}(F_k + F_k^\dagger)\rho\right) \quad \forall \rho\geq 0, \operatorname{tr}(\rho)=1.$

$\Rightarrow$ Assume from now on that $F_k = F_k^\dagger$.

iv) $1 = \sum p_k(\rho) = \operatorname{tr}\left((\sum F_k)\rho\right) \quad \forall \rho \geq 0$

and $F_k = F_k^\dagger \Rightarrow \sum F_k = I.$

v) $0 \leq p_k(\rho) = \operatorname{tr}(F_k\rho) \Rightarrow F_k \geq 0$

(otherwise $F_k$ has a negative eigenvalue

$\lambda < 0$, $\lambda |\phi\rangle = F_n |\phi\rangle$, and then

$$\mathrm{tr}\left(F_n |\phi\rangle\langle\phi|\right) = \lambda < 0 \text{ ↯ }$$ )

<u>Thus:</u> POVM measurement is the most general linear measurement on density matrices.