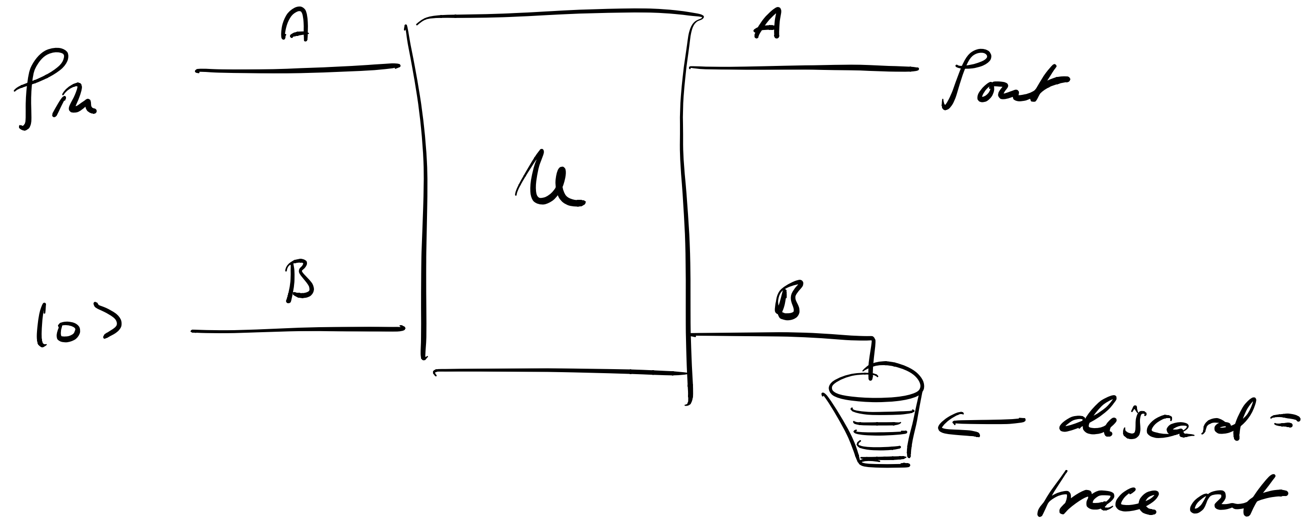
5. General evolution: Completely positive maps

What is the most general physical evolution of density matrices (a "superoperator")?

Same idea as for measurement – add ancillas:



... but now ancilla is simply discarded.

Analyze:

$$\rho \mapsto \mathcal{E}(\rho) = \mathrm{tr}_B\left[U \left(\rho \otimes |0\rangle_B\langle 0|_B \right) U^\dagger \right]$$

$$= \sum_n \langle n|_B \, U |0\rangle_B \; \rho \; \langle 0|_B U^\dagger |n\rangle_B$$

$$= \sum_n M_n \rho M_n^\dagger$$

with $M_n := {}_B\langle n| U |0\rangle_B$ (as for POVM).

Properties of M_n: As before, $\sum M_n^\dagger M_n = I$.

(Note: We can write the trace in a different basis $|\tilde{m}\rangle := \sum v_{nm} |n\rangle$, (v_{nm}) unitary

$\Longrightarrow \tilde{M}_m = \sum \overline{v_{nm}} M_n$ represents same evolution (cf. other ambiguities!)).

Definition (Kraus representation):

We call $\mathcal{E}(\rho) = \sum M_n \rho M_n^\dagger$, $\sum M_n^\dagger M_n = I$, the Kraus representation of $\mathcal{E}$.
The M_n are called Kraus operators.

(Note: Not all maps have a Kraus representation. But we will see that all physical maps have a Kraus representation.)

(Note: As discussed above, the Kraus rep. is not unique.)

Relation to POVM: Any such map can be understood as a POVM measurement where we discard the meas. outcome. In particular:

Relation to unitary + ancilla: Any map $\mathcal{E}$ with a Kraus form can be realized by adding an ancilla, evolving both, and discarding the ancilla. ("Stinespring dilation of $\mathcal{E}$")

Is this the most general physical map?

Minimal conditions on physical maps:

i) linear: $\mathcal{E}(\rho + \lambda\sigma) = \mathcal{E}(\rho) + \lambda\,\mathcal{E}(\sigma)$.
(required for ensemble interpretation)

ii) trace-preserving: $\mathrm{tr}(\mathcal{E}(\rho)) = \mathrm{tr}(\rho)$
(preserves probabilities)

iii) positive: $\rho \geq 0 \Rightarrow \mathcal{E}(\rho) \geq 0$.

(ii + iii $\Longleftrightarrow$ maps density matrices to density matrices)

Is this sufficient?

NO!

$\mathcal{E}$ should still be physical even if it acts on part of a larger system, i.e.,

$\mathcal{E}_A \otimes \mathcal{I}_B$ should still satisfy (i)–(iii).

(i), (ii) are implied by the above. But we get a new condition:

minimal conds for phys. maps (cont'd):

(iv) complete positivity:

For any dimension d_B of B,

$$\rho_{AB} \geq 0 \Longrightarrow (\mathcal{E}_A \otimes \mathcal{I}_B)(\rho_{AB}) \geq 0.$$

(Note: The map $\mathcal{E} \otimes \mathcal{I}$ is yet again defined through linearity, i.e. $(\mathcal{E} \otimes \mathcal{I})(N \otimes M) = \mathcal{E}(N) \otimes \mathcal{I}(M)$, + linearity).

Definition: We call a map $\mathcal{E}: \rho \mapsto \mathcal{E}(\rho)$ satisfying the conditions (i)–(iv) above a completely positive trace-preserving (CPTP) map, or a quantum channel.

Are there maps which are positive ((i)–(iii)) but not completely positive?

YES! E.g. "transposition map"

$$\mathcal{E}(\rho) = \rho^T$$

$$(\mathcal{E} \otimes I)(\rho_{AB}) =: \rho_{AB}^{T_A} \quad \text{"partial transpose"}$$

Consider action of $\mathcal{E} \otimes I$ on $|\Omega\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$:

– since $(|i\rangle\langle j|)^T = |j\rangle\langle i| \Rightarrow (|ik\rangle\langle jl|)^{T_A} = |jk\rangle\langle il|$ –

$$(\mathcal{E}\otimes I)(|\Omega\rangle\langle\Omega|) = (|\Omega\rangle\langle\Omega|)^{T_A}$$

$$= \frac{1}{2}\left[|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right]$$

$$= \frac{1}{2}\left[|00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11|\right]$$

$$= \frac{1}{2}\begin{pmatrix} 1 & & & 0 \\ & 0 & 1 & \\ & 1 & 0 & \\ 0 & & & 1 \end{pmatrix} \not\geq 0 \quad !$$

Note: Positive but not completely positive maps are important tools to detect entanglement, since they satisfy $(\mathcal{E}\otimes I)(\rho) \geq 0$ for any unentangled state, i.e.: $(\mathcal{E}\otimes I)(\rho) \not\geq 0 \Rightarrow \rho$ entangled!

(→ Chapter III!)

Lemma: Any $\mathcal{E}$ in Kraus form is CPTP.

Proof: Either by construction, or by direct inspection of

$$(\mathcal{E}\otimes I)(\rho) = \sum \underbrace{(\Pi_u\otimes I)\rho(\Pi_u\otimes I)^\dagger}_{\geq 0} \geq 0 \qquad \boxtimes$$

Can conversely all CPTP maps be written in Kraus form? If yes, how can we obtain the Kraus operators?

Key tool: The Choi-Jamiołkowski isomorphism.

Theorem (Choi-Jamiołkowski isomorphism)

reminder: $B(X) =$ lin. maps on X.

Let $\mathcal{C} := \{\mathcal{E} \mid \mathcal{E} \text{ CPTP}\} \subset B(B(\mathbb{C}^d))$ the space of all CPTP maps on the density operators on $\mathbb{C}^d$, and

$\mathcal{J} := \{\sigma_{AB} \mid \sigma_{AB} \geq 0, \operatorname{tr}_A(\sigma_{AB}) = \frac{1}{d} I\} \subset B(\mathbb{C}^d \otimes \mathbb{C}^d)$

the space of all bipartite states with $\operatorname{tr}_A(\sigma_{AB}) = \frac{1}{d} I$.

Then, the map

$$\hat{X} : B(B(\mathbb{C}^d)) \longrightarrow B(\mathbb{C}^d \otimes \mathbb{C}^d)$$

$$\mathcal{E} \longmapsto \sigma_{AB} = (\mathcal{E}_A \otimes I_B)(|\Omega\rangle\langle\Omega|),$$

$$|\Omega\rangle := \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i,i\rangle$$

defines an isomorphism between $\mathcal{C}$ and $\mathcal{S}$, the Choi-Jamiołkowski isomorphism, with σ_{AB} the Choi state of $\mathcal{E}$. The inverse map is

$$\hat{\mathcal{Y}} : \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d) \longrightarrow \mathcal{B}(\mathcal{B}(\mathbb{C}^d))$$

$$\sigma_{AB} \longmapsto \mathcal{F},$$

$$\text{where } \mathcal{F}(\rho) = d \cdot \operatorname{tr}_B\left[\sigma_{AB} \cdot \left(\mathbb{I}_A \otimes \rho^T\right)\right].$$

(Note: A physical interpretation of $\hat{\mathcal{X}}$, $\hat{\mathcal{Y}}$, and the theorem will be given in Chapter III.)

Proof: We need to show:

(i) $\hat{\mathcal{Y}} \circ \hat{\mathcal{X}} = \mathbb{I}$

(ii) $\hat{\mathcal{X}} \circ \hat{\mathcal{Y}} = \mathbb{I}$

(iii) $\operatorname{Im}(\hat{\mathcal{X}}|_{\mathcal{C}}) = \{\hat{\mathcal{X}}(\mathcal{E}) \mid \mathcal{E} \in \mathcal{C}\} \subset \mathcal{S}$

(iv) $\operatorname{Im}(\hat{\mathcal{Y}}|_{\mathcal{S}}) \subset \mathcal{C}$.

Together, (i)–(iv) imply

a) (i) $\Longrightarrow$ $\hat{X}$ injective

b) $s \in \mathcal{S} \Longrightarrow c := \hat{Y}s \overset{(iv)}{\in} \mathcal{C}$ & $\hat{X}c \overset{(ii)}{=} s$

$$\left.\begin{array}{l} \Longrightarrow \mathrm{Im}\, \hat{X}|_{\mathcal{C}} \supset \mathcal{S} \\ \text{and from (iii):} \\ \mathrm{Im}\, \hat{X}|_{\mathcal{C}} \subset \mathcal{S} \end{array}\right\} \Longrightarrow \mathrm{Im}\, \hat{X}|_{\mathcal{C}} = \mathcal{S}$$

$$\Longrightarrow \hat{X}|_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{S} = \mathrm{Im}\, \hat{X}|_{\mathcal{C}}$$

is a linear bijection!

Proof of (i): $\hat{Y} \circ \hat{X} = \mathbb{I}$:

Need to show $\hat{Y}(\hat{X}(\mathcal{E})) = \mathcal{E}$ for all $\mathcal{E} \in \mathcal{B}(\mathcal{B}(\mathbb{C}^d))$,

any $\rho \in \mathcal{B}(\mathbb{C}^d)$

$$\hat{Y}(\underbrace{\hat{X}(\mathcal{E})}_{\equiv \sigma_{AB}})(\rho) = d \cdot \mathrm{tr}_B\big[\underbrace{\hat{X}(\mathcal{E})}_{\equiv \sigma_{AB}} \cdot (\mathbb{I}_A \otimes \rho^T)\big]$$

$$= d \cdot \frac{1}{d} \sum_{ij} \mathrm{tr}_B\Big[\underbrace{\big((\mathcal{E} \otimes \mathbb{I}_B)(|i\rangle\langle j| \otimes |i\rangle\langle j|)\big)}_{\mathcal{E}(|i\rangle\langle j|) \otimes |i\rangle\langle j|} (\mathbb{I}_A \otimes \rho^T)\Big]$$

$$= \sum_{ij} \mathcal{E}(|i\rangle\langle j|) \cdot \underbrace{\mathrm{tr}\left[|i\rangle\langle j|\rho^T\right]}_{=\langle j|\rho^T|i\rangle = \rho_{ij}}$$

$$= \mathcal{E}\Big(\sum_{ij} \rho_{ij} |i\rangle\langle j|\Big)$$

$$= \underline{\mathcal{E}(\rho)}.$$

$\underline{\text{I.e.:}}$ $\hat{y}(\hat{x}(\mathcal{E}))(\rho) = \mathcal{E}(\rho) \quad \forall \rho, \mathcal{E}$

$$\Rightarrow \hat{y}(\hat{x}(\mathcal{E})) = \mathcal{E} \quad \forall \mathcal{E} \qquad \blacksquare$$

<u>Proof of (ii): $\hat{x} \circ \hat{y} = I$.</u>

For any $\sigma_{AB} \in B(\mathbb{C}^d \otimes \mathbb{C}^d)$,

$$\underbrace{\hat{x}(\hat{y}(\sigma_{AB}))}_{\equiv \mathcal{F}} = (\underbrace{\hat{y}(\sigma_{AB})}_{\equiv \mathcal{F}_A} \otimes I_B)(|\Omega\rangle\langle\Omega|)$$

$$= \frac{1}{d} \sum_{ij} \underbrace{\hat{y}(\sigma_{AB})}_{\equiv \mathcal{F}_A}(|i\rangle\langle j|_A) \otimes |i\rangle\langle j|_B$$

$$= \frac{1}{d} \sum_{ij} d \cdot \mathrm{tr}_B\left[\sigma_{AB} \cdot (I_A \otimes |i\rangle\langle j|)_B^T\right] \otimes |i\rangle\langle j|_B$$

$$= \sum_{ij} \langle i|_B \, \sigma_{AB} \, |j\rangle_B \otimes |i\rangle\langle j|_B$$

$$= \sigma_{AB} .$$

$$\Rightarrow \hat{X} \circ \hat{Y} = I . \qquad \blacksquare$$

<u>Proof of (iii): Im $\hat{X}|_{\mathcal{C}} \subset \mathcal{J}$.</u>

Let $\mathcal{E} \in \mathcal{C}$, i.e., $\mathcal{E}$ is a CPTP map.

Then, $\sigma_{AB} := \hat{X}(\mathcal{E}) = (\mathcal{E}_A \otimes I_B)(|\Omega\rangle\langle\Omega|) \geq 0$

since $\mathcal{E}$ is completely positive.

Further, $\underline{tr_A(\sigma_{AB})} = \frac{1}{d} \sum tr_A\left[(\mathcal{E}_A \otimes I_B)(|ii\rangle\langle jj|)\right]$

$$= \frac{1}{d} \sum \underbrace{tr\left[\mathcal{E}(|i\rangle\langle j|)\right]}_{= tr\, |i\rangle\langle j| = \delta_{ij}} = \underline{\frac{1}{d} I_B} .$$

$\mathcal{E}$ trace preserving

$$\Rightarrow \sigma_{AB} = \hat{X}(\mathcal{E}) \in \mathcal{J} \quad \forall \mathcal{E} \in \mathcal{C} .$$

Proof of (iv): $\operatorname{Im} \hat{\mathcal{J}}|_{\mathcal{S}} \subset \mathcal{C}$.

Let $\sigma_{AB} \in \mathcal{S}$. Write $\sigma_{AB} = \sum_k |\tilde{\psi}_k\rangle\langle\tilde{\psi}_k|$ un-normalized!

can be any decomposition – e.g. eigenvalue dec.

Expand $|\tilde{\psi}_k\rangle = \sum \frac{1}{\sqrt{d}} m_k^{ji} |j\rangle|i\rangle$

$$= \frac{1}{\sqrt{d}} \sum_i M_k |i\rangle|i\rangle$$

$$= (M_k \otimes I)|\Omega\rangle,$$

where M_k has entries $(M_k)_{ji} = m_k^{ji}$.

Then, $\sigma_{AB} = \sum_k (M_k \otimes I)|\Omega\rangle\langle\Omega|(M_k \otimes I)^\dagger$.

We want to show: $\hat{\mathcal{J}}(\sigma_{AB}) \equiv \mathcal{F}$ is CPTP:

$$\underbrace{\hat{\mathcal{J}}(\sigma_{AB})(\rho)}_{\equiv \mathcal{F}(\rho)} = d \cdot \operatorname{tr}_B\left[\sigma_{AB}(I \otimes \rho^T)\right]$$

$$= d \cdot \operatorname{tr}_B\left[\sum_k (M_k \otimes I)|\Omega\rangle\langle\Omega|(M_k \otimes I)^\dagger (I \otimes \rho^T)\right]$$

$$= d \cdot \sum_k \Pi_k \underbrace{\mathrm{tr}_B \left[(|\Omega\rangle\langle\Omega|)(I \otimes \rho^T) \right]}_{= \frac{1}{d} \sum_{ij} |i\rangle\langle j|_A \underbrace{\mathrm{tr}\left[|i\rangle\langle j| \rho^T \right]}_{= \rho_{ij}} \; = \frac{1}{d} \rho} \Pi_k^\dagger$$

$$= \sum_k \Pi_k \rho \Pi_k^\dagger .$$

Moreover, $\frac{1}{d} I = \mathrm{tr}_A \sigma_{AB}$

$$= \mathrm{tr}_A \left[\sum_k (\Pi_k \otimes I) |\Omega\rangle\langle\Omega| (\Pi_k \otimes I)^\dagger \right]$$

$$= \sum_k \mathrm{tr}_A \left[(\Pi_k^\dagger \Pi_k \otimes I) |\Omega\rangle\langle\Omega| \right]$$

$$= \frac{1}{d} \sum_{ijk} \underbrace{\mathrm{tr}\left(\Pi_k^\dagger \Pi_k |i\rangle\langle j| \right)}_{= \langle j| \Pi_k^\dagger \Pi_k |i\rangle} |i\rangle\langle j|$$

$$\Rightarrow \sum_k \langle j| \Pi_k^\dagger \Pi_k |i\rangle = \delta_{ij}$$

$$\Rightarrow \sum_k \Pi_k^\dagger \Pi_k = I .$$

Thus, $\hat{\mathcal{G}}(\sigma_{AB})(\rho) = \sum \Pi_k^\dagger \rho \Pi_k$ w/ $\sum \Pi_k^\dagger \Pi_k = I$;

i.e., $\hat{\mathcal{G}}(\sigma_{AB})$ has a Kraus representation

and is thus a CPTP map, $\hat{\mathcal{G}}(\sigma_{AB}) \in \mathcal{S}_B$

$\blacksquare$

<u>Note:</u> The isomorphism still holds if we drop trace preserving from $\mathcal{C}$ and $\mathrm{tr}_A \sigma_{AB} = \frac{1}{d} I$ from $\mathcal{S}$, respectively.

<u>Corollary (from the proof of (iv)):</u>

All CPTP maps are of Kraus form, and can thus be realized with a Stinespring dilation (i.e., add ancilla + unitary + tracing).

Moreover, the Kraus operators Π_k can be obtained from the Choi state σ_{AB} by writing

$\sigma_{AB} = \sum |\tilde{\varphi}_k\rangle\langle\tilde{\varphi}_k|$, and $|\tilde{\varphi}_k\rangle = (\Pi_k \otimes I)|\Omega\rangle$.