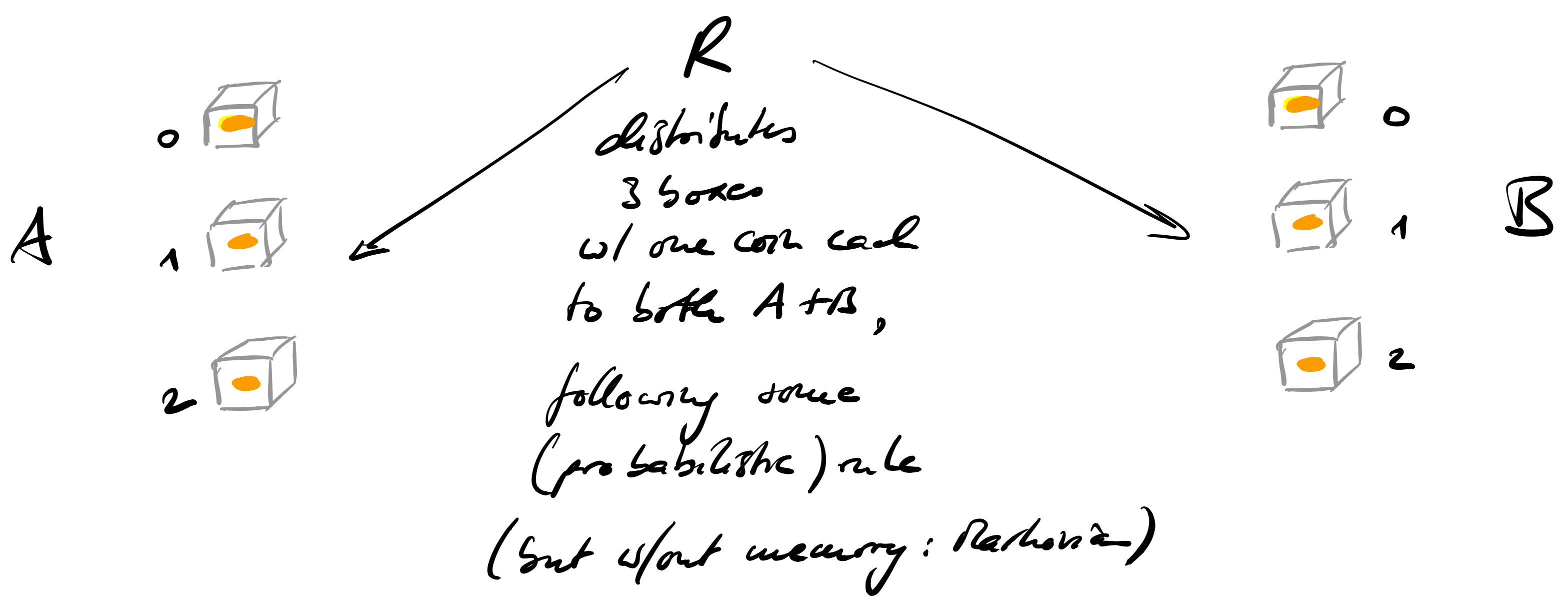
2. Bell inequalities

"How non-classical are entangled states?"

a) The Bell inequality

Consider the following game played by Alice + Bob with coins, prepared by a referee R:



(i)

- A & B play many rounds w/ the referee. In each round, A & B each get 3 coins in closed boxes (labelled 0,1,2), prepared according to some rule (deterministic or random, but the same indep. rule in every rd.) by R.
- In each rd., A & B can look at only <u>one coin each</u> ($x=0,1,2$; $y=0,1,2$). We denote heads $=+1$

and tails $= -1$; and the obtained results by $a_x = \pm 1$; $b_y = \pm 1$.

- After that, the boxes are collected by the referee, and a new round starts.

(ii) By repeatedly measuring the same box, $x = y$, A & B observe: They always get the <u>same outcome</u>, i.e.

$$\boxed{a_x = b_x}$$

(iii) A & B are smart: They can use this to cheat the referee and obtain the value of <u>two coins</u> in <u>a single round</u>.

<u>Idea:</u> A checks coin x, B checks coin $y = x' \neq x$.

Then, since $a_{x'} = b_{x'}$, they know <u>a_x and $a_{x'}$</u> in the same round!

This <u>clearly works</u> in a classical scenario (i.e., with coins – we will formalize this later).

Consequence: In a class. world, A & B can use this to estimate

$$p(a_x = a_{x'}) \approx \frac{N(a_x = a_{x'})}{N_{tot}} \quad \forall x, x'!$$

exact as $N \to \infty$!

probability

# of rounds where $a_x = a_{x'}$

total rounds

Then,

$$p(a_0 = a_1) + p(a_1 = a_2) + p(a_2 = a_0) \geq 1,$$

$$\frac{N(a_0 = a_1)}{N_{tot}} + \frac{N(a_1 = a_2)}{N_{tot}} + \frac{N(a_2 = a_0)}{N_{tot}}$$

since in each round, <u>at least</u> two coins must be equal (or all 3)!

Using that (classically) $a_x = b_x$:

$$\rightarrow \boxed{p(a_0 = b_1) + p(a_1 = b_2) + p(a_2 = b_0) \geq 1} \quad (*)$$

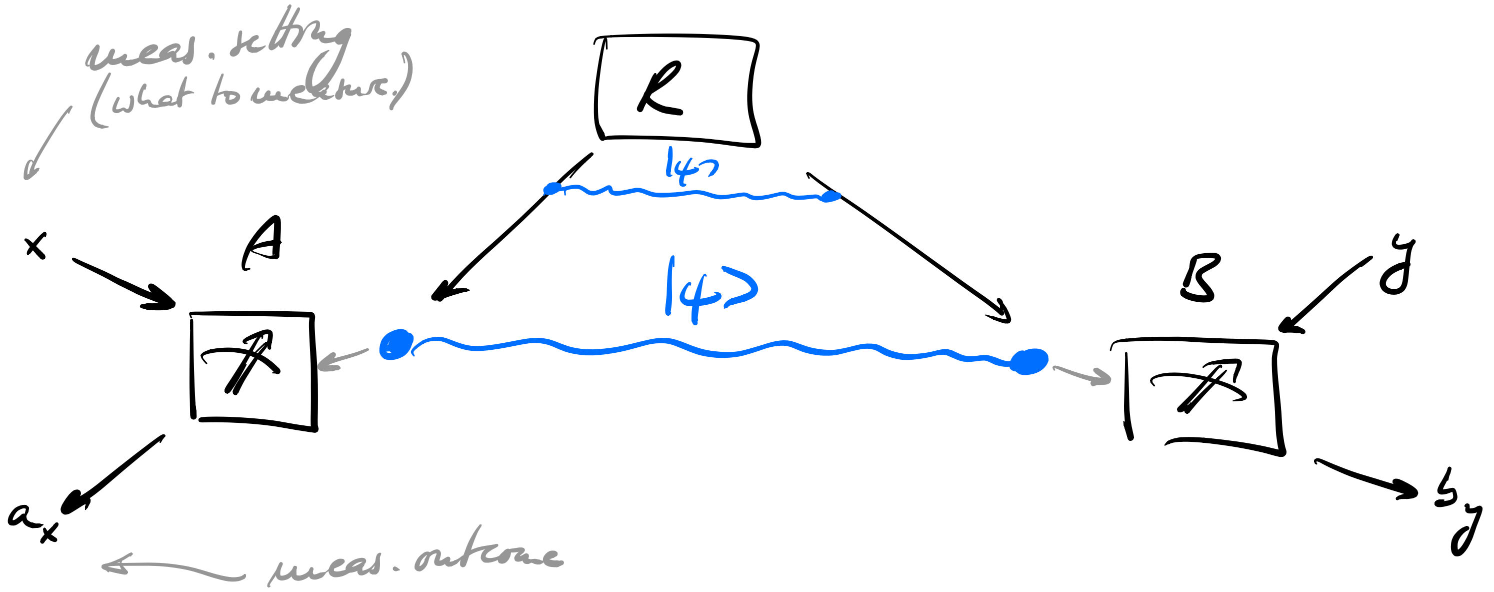
is satisfied for any classical theory (which satisfies $a_x = b_x \ \forall x$).

(*) is called <u>Bell inequality</u>!

(Note: Bell inequalities are inequalities satisfied by classical theories, and have a priori nothing to do with quantum theory!)

But: In a suitable quantum mechanical version of the game, ⊛ is violated!

Quantum version of the game:



- R distributes bipartite state $|\psi\rangle$.
- A & B perform a measurement dep. on x/y, w/ outcome a_x/b_y.

(I.e.: Which box to open = which meas. to perform — A & B can always only do <u>one</u> meas. each on original state).

<u>Setup:</u> • $|\psi\rangle = |\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

• A & B do projective measurement along axes $\vec{u}_x$ and $\vec{m}_y$, i.e. they measure the operators $\vec{u}_x \cdot \vec{\sigma}^A$ and $\vec{m}_y \cdot \vec{\sigma}^B$.

• We will specify $\vec{u}_x$ and $\vec{m}_y$ later on.

It holds that $(\underbrace{\vec{\sigma}^A}_{\equiv \vec{\sigma}_A \otimes I_B} + \underbrace{\vec{\sigma}^B}_{\equiv I_A \otimes \vec{\sigma}^B})|\psi^-\rangle = 0$

(i.e. $(\sigma^A_\alpha + \sigma^B_\alpha)|\psi^-\rangle = 0 \quad \forall \alpha = x,y,z$;

by direct inspection, or since $|\psi^-\rangle$ has spin 0).

$\Rightarrow \langle\psi^-|(\vec{\sigma}^A\cdot\vec{u})\underbrace{(\vec{\sigma}^B\cdot\vec{m})|\psi^-\rangle}_{= -\vec{\sigma}^A\cdot\vec{m}|\psi^-\rangle \text{ from the above}} =$

$= -\langle\psi^-|(\vec{\sigma}^A\cdot\vec{u})(\vec{\sigma}^A\cdot\vec{m})|\psi^-\rangle$

$$= -\sum_{kl} u_k m_l \underbrace{\langle\psi^-|\sigma_k^A \sigma_l^A|\psi^-\rangle}_{\substack{= \mathrm{tr}[\rho_A \sigma_k^A \sigma_l^A] \\ \rho_A = \frac{1}{2}I, \text{ e.g. from Schmidt dec.} \\ = \frac{1}{2}\mathrm{tr}[\sigma_k^A \sigma_l^A] = \delta_{kl}}}$$

$$= -\sum_k u_k m_k = -\vec{u}\cdot\vec{m} = -\cos\theta$$

angle between $\vec{u}$ and $\vec{m}$.

Measurement along $\vec{u}$: Meas. operators (projectors)

$$E_{\pm 1}(\vec{u}) = \frac{1}{2}\left(I \pm \vec{u}\cdot\vec{\sigma}\right)$$

Let $p(a,b)$, $a,b = \pm 1$ denote prob. to get outcomes a & b, respectively, for A & B. Then,

$$p(\pm 1, \pm 1) = \langle\psi^-|\, E_\pm^A(\vec{u})\, E_\pm^B(\vec{m})\,|\psi^-\rangle$$

$$= \frac{1}{4}\langle\psi^-|\underbrace{I}_{=1} \pm \underbrace{\vec{u}\cdot\vec{\sigma}^A}_{\substack{=0 \text{ since} \\ \rho_A = \frac{1}{2}I}} \pm \underbrace{\vec{m}\cdot\vec{\sigma}^B}_{=0} + \underbrace{(\vec{u}\cdot\vec{\sigma}^A)(\vec{m}\cdot\vec{\sigma}^B)}_{=-\cos\theta}|\psi^-\rangle$$

$$= \frac{1}{4}(1-\cos\theta).$$

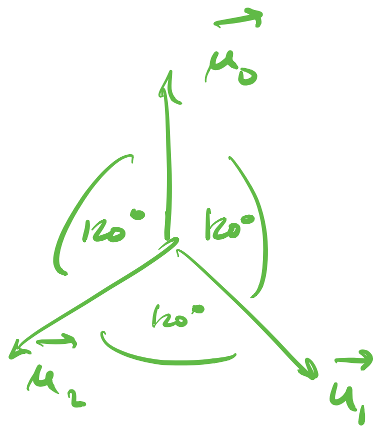
and

$$p(\pm 1, \mp 1) = \frac{1}{4}(1+\cos\theta).$$

$$\Rightarrow \text{prob}(a=b) = \frac{1}{2}(1-\cos\theta)$$

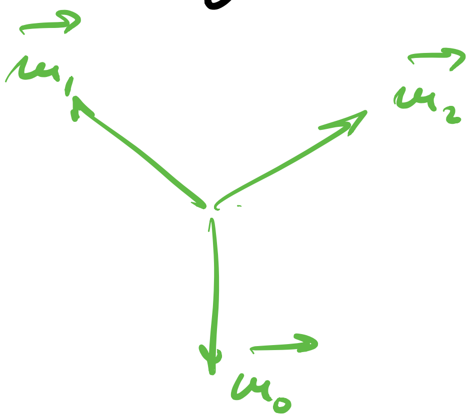
$$\text{prob}(a\neq b) = \frac{1}{2}(1+\cos\theta)$$

Now let A choose measurements $\vec{m}_0, \vec{m}_1, \vec{m}_2$



in the XZ-plane,

and B along $\vec{m}_x = -\vec{u}_x$:



Then:

- $x = y$: $p(a_x = b_y) = \frac{1}{2}(1 - \cos 180°) = 1$

$\Rightarrow$ A & B always get <u>same result</u> when they measure "the same coin"

- $x \neq y$: $p(a_x = b_y) = \frac{1}{2}(1 - \underbrace{\cos(\pm 60°)}_{= 1/2}) = \frac{1}{4}$

$$\Rightarrow \quad p(a_0 = b_1) + p(a_1 = b_2) + p(a_2 = b_0) = \frac{3}{4} < 1$$

(while Bell ineq. stated ≥ 1 for class. theories!)

<u>**Bell inequality violated!**</u>

Formally, what were the <u>assumptions</u> of our <u>classical theory</u>?

① <u>Realism</u>: Outcomes of measurements are "elements of reality" – i.e., they have pre-determined values even prior to measurement.

② **Locality:** A & B's boxes cannot communicate once distributed.

⇒ Quantum mechanical predictions are **incompatible** with any **local and realistic** theory — we need to give up either locality or realism.

Bell inequalities can be used to certify that a system behaves quantum mechanically (i.e. non-classically): If we measure a violation of the Bell inequality — note that $p(a_x = b_y)$ can be estimated reliably by repeated measurements — we know that the system cannot be described classically and must thus be quantum mechanical.

b) The CHSH inequality

Bell's inequality has two downsides:

i) We first need to separately test that $a_x = b_x$

ii) We need 3 different measurement settings

— maybe with only 2 settings, everything can be modelled classically?

Consider setting with 2 measurements: $x = 0,1$; $y = 0,1$, again with outcomes $a_x, b_y = \pm 1$ (minimal setting).

Since $a_x = \pm 1$, $b_y = \pm 1$:

$$C = \underbrace{(a_0 + a_1)}\, b_0 + \underbrace{(a_0 - a_1)}\, b_1 = \pm 2$$

one of these must be 0, the other is ± 2.

average over many iterations (prob. distr.)

$$\Rightarrow |\langle C \rangle| \leq \langle |C| \rangle = 2$$

$$\boxed{\left| \langle a_0 b_0 \rangle + \langle a_1 b_0 \rangle + \langle a_0 b_1 \rangle - \langle a_1 b_1 \rangle \right| \leq 2}$$

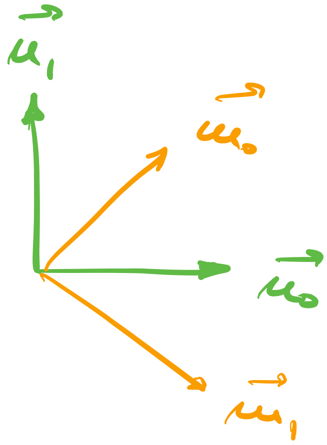
"**CHSH inequality**" (Clauser, Horne, Shimony, Holt)

Violation of CHSH inequality in quantum theory:

Take $|\psi\rangle = |\psi^-\rangle$

$a_x \longleftrightarrow \vec{u}_x \cdot \vec{\sigma}^A$

$b_y \longleftrightarrow \vec{w}_y \cdot \vec{\sigma}^B$



XZ plane

Have seen:

$$\langle a_x b_y \rangle = -\cos\theta$$

$$\Longrightarrow \langle a_0 b_0 \rangle = \langle a_0 b_1 \rangle = \langle a_1 b_0 \rangle = -\frac{1}{\sqrt{2}}$$

$$\langle a_1 b_1 \rangle = +\frac{1}{\sqrt{2}}$$

$$\Longrightarrow \left| \langle a_0 b_0 \rangle + \langle a_1 b_0 \rangle + \langle a_0 b_1 \rangle - \langle a_1 b_1 \rangle \right| = 2\sqrt{2} > 2 \; !$$

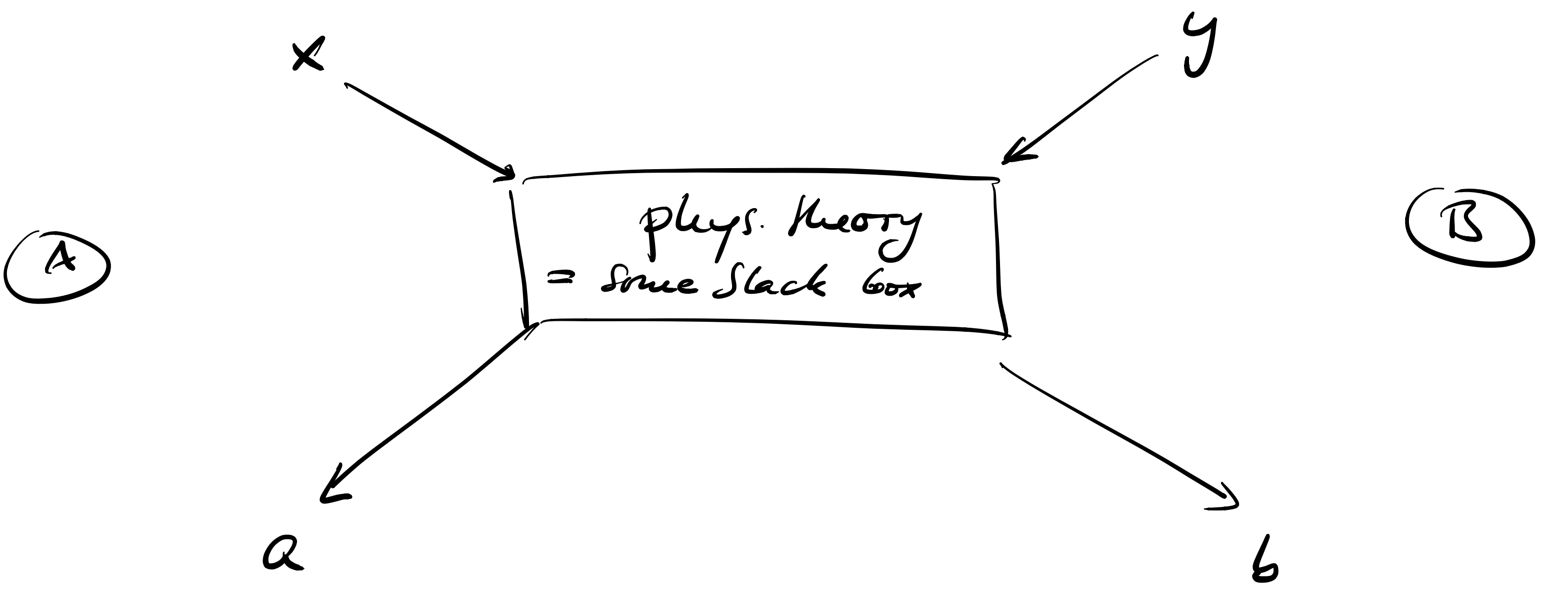
CHSH inequality violated!

Note: This violation is optimal (maximal) within QM.

(But: With a general local theory, $|\langle C \rangle| = 4$ can be obtained: QM is more restrictive than general local theories. (→ Homework))

c) Formal setup and local hidden variable theories

Formal setup for physical theories in bipartite setting:



A: input x (meas. setting),
output a (meas. result)

B: input y, output b.

Any physical theory is characterized by a conditional probability distribution

$$P(a,b|x,y)$$

to obtain a & b given x & y, where

$$\sum_{a,b} P(a,b|x,y) = 1 \quad \forall x,y.$$

<u>Question</u>: Which $P(a,b|x,y)$ are consistent with a given physical theory?

Classical physics:

"<u>Local hidden-variable (LHV) model</u>":

All outcomes are pre-determined by some "hidden variable" λ, which is chosen according to some distribution & give to A&B, who act independently (i.e., locally) conditioned on λ.

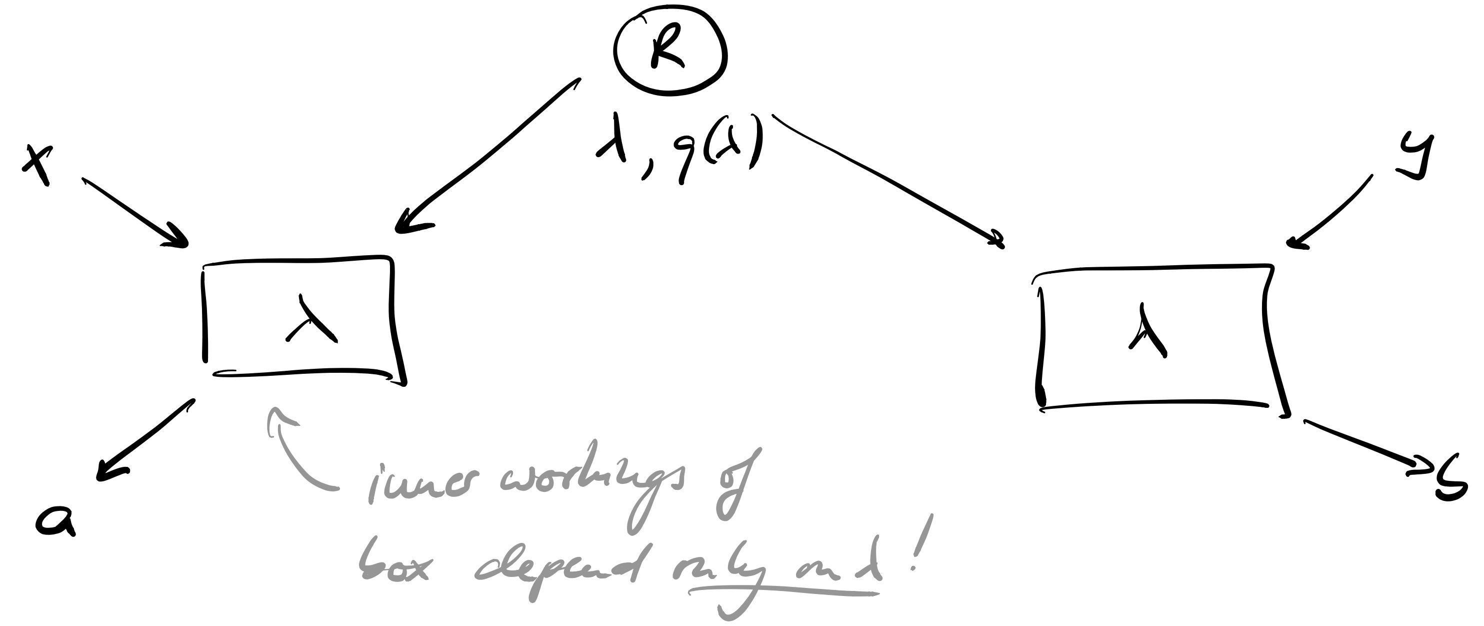
("<u>Local realism</u>": Outcomes exist indep. of meas. – realism – and no (faster-than-light) communication between A & B — locality.)

i.e.:

$$\circledast \quad P(a,b|x,y) = \sum_\lambda q(\lambda)\, P^A_\lambda(a|x)\, P^B_\lambda(b|y)$$

prob. distr. over λ

can be made deterministic by putting all randomness in λ & $q(\lambda)$.



How have we been using the LHV form $\circledast$ above in the derivation of Bell-type inequalities?

E.g. CHSH:

$$\langle C\rangle = \langle a_0 b_0\rangle + \langle a_1 b_0\rangle + \langle a_0 b_1\rangle - \langle a_1 b_1\rangle$$

$$= \sum_{x,y} (-1)^{xy} \langle a_x b_y \rangle$$

$$= \sum_{x,y} (-1)^{xy} \Big[\sum_{a_x, b_y} a_x b_y \; P(a_x, b_y | x, y) \Big]$$

$$P(a_x, b_y | x, y) = \sum q(\lambda)\, P_\lambda^A(a_x|x)\, P_\lambda^B(b_y|y)$$

$$= \sum_\lambda q(\lambda) \underbrace{\sum_{x,y} (-1)^{xy} \Big[\sum_{a_x, b_y} a_x b_y \; P_\lambda^A(a_x|x)\, P_\lambda^B(b_y|y) \Big]}_{=:\, E_\lambda}$$

Then, $|\langle C\rangle| \leq \sum_\lambda q(\lambda)\, |E_\lambda|$, and

$$|E_\lambda| = \Big| \sum_{x,y} (-1)^{xy} \underbrace{\Big(\sum_{a_x} a_x \; P_\lambda^A(a_x|x) \Big)}_{=\langle a_x\rangle_\lambda} \underbrace{\Big(\sum_{b_y} b_y \; P_\lambda^B(b_y|y) \Big)}_{=\langle b_y\rangle_\lambda} \Big|$$

$$= \Big| \big(\langle a_0\rangle_\lambda + \langle a_1\rangle_\lambda \big) \langle b_0\rangle_\lambda + \big(\langle a_0\rangle_\lambda - \langle a_1\rangle_\lambda \big) \langle b_1\rangle_\lambda \Big|$$

$$\leq \left| \langle a_0 \rangle_\lambda + \langle a_1 \rangle_\lambda \right| \underbrace{\left| \langle b_0 \rangle_\lambda \right|}_{\leq 1} + \left| \langle a_0 \rangle_\lambda - \langle a_1 \rangle_\lambda \right| \underbrace{\left| \langle b_1 \rangle_\lambda \right|}_{\leq 1}$$

$$\leq \left| \langle a_0 \rangle_\lambda + \langle a_1 \rangle_\lambda \right| + \left| \langle a_0 \rangle_\lambda - \langle a_1 \rangle_\lambda \right|$$

$$\leq 2 \max \left\{ \left| \langle a_0 \rangle_\lambda \right|, \left| \langle a_1 \rangle_\lambda \right| \right\} \leq 2.$$

Where have we used the LHV condition ⊛ ?

→ In factorizing

$$\langle a_x b_y \rangle_\lambda = \sum_{a_x, b_y} a_x b_y \; P(a_x, b_y | x, y)$$

$$= \Big(\sum_{a_x} a_x \; P_\lambda^A(a_x|x) \Big) \Big(\sum_{b_y} b_y \; P_\lambda^B(b_y|y) \Big)$$

$$= \langle a_x \rangle_\lambda \langle b_y \rangle_\lambda$$

– otherwise, it does not make sense to talk about $\langle a_x \rangle$ indep. of the value of y, and $\langle a_0 b_0 + a_0 b_1 + a_1 b_0 - a_1 b_1 \rangle$ cannot be factorized (either as expectation values $\langle a_i \rangle \langle b_j \rangle$, or classical variables – coins – which is in essence the same.)