5. Entanglement of mixed states

a) Introduction

When is a mixed bipartite state ρ_{AB} entangled?

Different possible definitions:

(i) If ρ_{AB} cannot be created by LOCC.

(ii) If we can extract ebits $|\phi^+\rangle$ from ρ_{AB}.

(iii) If it helps us to do some task better in an LOCC setting (it is a "resource").

... any of these could be in a finite-copy or asymptotic setting!

Clearly, (ii) $\geq$ (iii) $\geq$ (i)

"$\geq$": is a stronger condition, i.e. satisfied by less states.

We use the weakest notion (i) to define entangled states.

Definition:

A bipartite state ρ_{AB} is called separable if it can be written as

$$(*) \quad \boxed{\rho_{AB} = \sum p_i \, \rho_i^A \otimes \rho_i^B} \quad \text{"separable state"}$$

for some $p_i \geq 0$, $\sum p_i = 1$;

$\rho_i^A \geq 0$, $\rho_i^B \geq 0$; $\mathrm{tr}\, \rho_i^A = \mathrm{tr}\, \rho_i^B = 1$

— i.e., ρ_{AB} can be prepared by LOCC.

If ρ_{AB} is not separable — i.e., it has no decomposition of the form $(*)$ — it is called entangled.

Given a state ρ_{AB}, how can we tell if it is entangled?

Problem: Given ρ_{AB}, need to check all decompositions

$$\rho_{AB} = \sum p_i \, \rho_i^{AB} = \sum p_i \, \rho_i^A \otimes \rho_i^B$$

(or $\rho_{AB} = \sum p_i |\psi_i^{AB}\rangle\langle\psi_i^{AB}|$) to find if there is a separable one. (Ambiguity in ensemble decomposition is an isometry, i.e., we need to optimize over isometries!)

$\longrightarrow$ <u>Difficult!</u>

(In fact, the general problem has been shown to be NP-hard in the dimension of the space.)

$\longrightarrow$ Need <u>partial solutions</u>, e.g. ways to certify a given state is entangled.

b) Entanglement witnesses

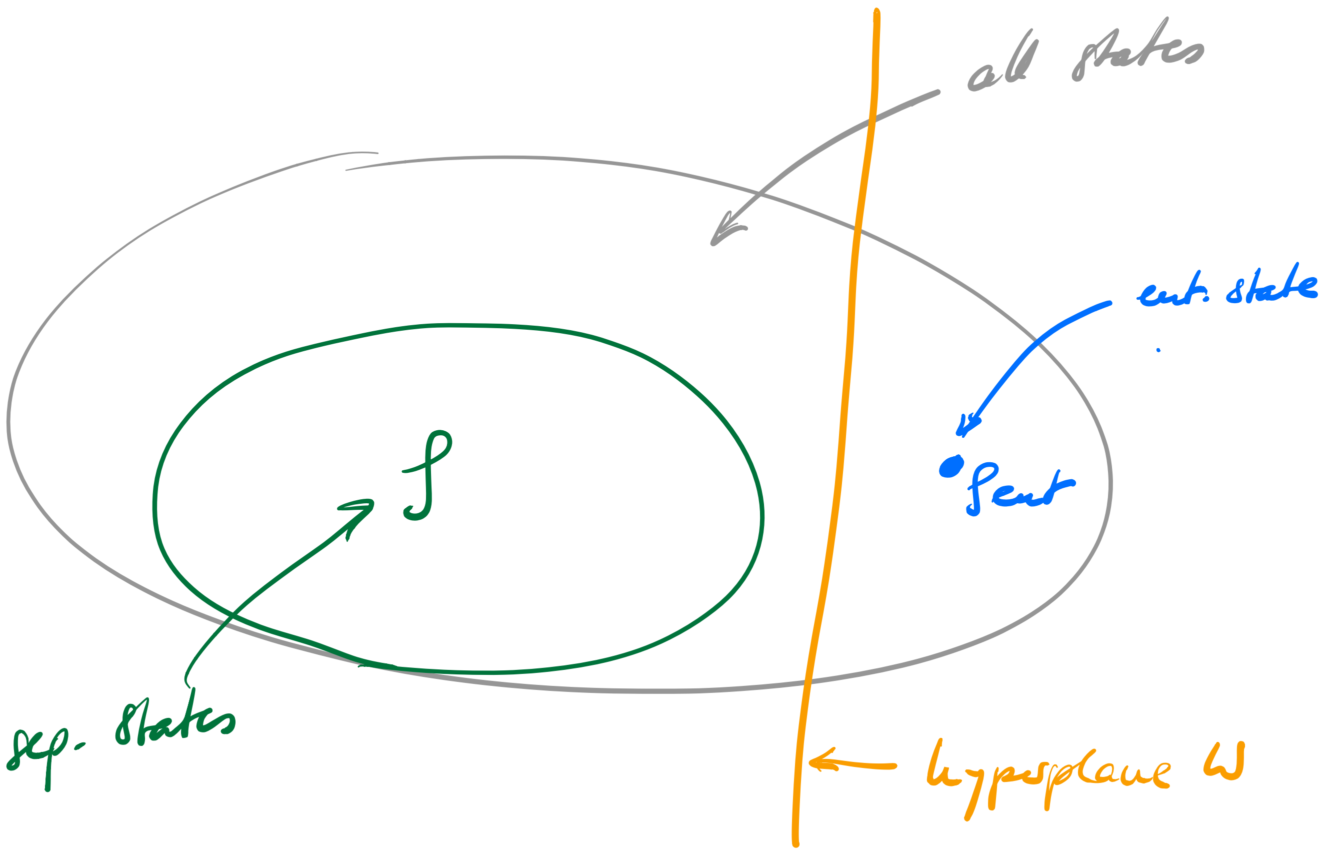
Structure of the set $\mathcal{S}$ of separable states:

Let $\rho = \sum_{i=1}^{N} p_i \rho_i^A \otimes \rho_i^B \in \mathcal{S}$, $\sigma = \sum_{j=1}^{M} q_j \sigma_j^A \otimes \sigma_j^B \in \mathcal{S}$

$\Rightarrow$ for $\lambda \in [0;1]$,

$$\underline{\lambda\rho + (1-\lambda)\sigma} = \lambda p_1 \rho_1^A \otimes \rho_1^B + \ldots + \lambda p_N \rho_N^A \otimes \rho_N^B + \ldots$$
$$+ (1-\lambda) q_1 \sigma_1^A \otimes \sigma_1^B + \ldots + (1-\lambda) q_M \sigma_M^A \otimes \sigma_M^B \,\underline{\in S}$$

$\Rightarrow$ the set S of separable states forms a <u>convex set</u>



For any state $\rho_{ent} \notin S$, we can find a <u>hyperplane</u> which separates ρ_{ent} from S.

More generally, we can construct hyperplanes s.th. all points on one side of the plane are entangled (but not the other way).

Any hyperplane is of the form $\mathrm{tr}[X\rho]+c=0$,

i.e., $\mathrm{tr}[\underbrace{(X+cI)}_{=:W}\rho]=\mathrm{tr}[W\rho]=0$.

We can choose $W=W^\dagger$ as we work on the space of hermitian matrices.

Then:

$\mathrm{tr}(\rho W)>0$: ρ left of hyperplane W

$\mathrm{tr}(\rho W)<0$: ρ right of hyperplane W

I.e. for a hyperplane as afore:

$$\rho \text{ separable} \Longrightarrow \mathrm{tr}(W\rho)\geq 0$$

and thus

$$\mathrm{tr}(W\rho)<0 \Longrightarrow \rho \text{ entangled.}$$

Definition (entanglement witness):

An operator $W = W^\dagger$ such that

$$\rho \text{ separable} \implies tr(W\rho) \geq 0$$

is called an **entanglement witness.**

Observation: Given an entanglement witness W,

$$tr(\rho W) < 0 \implies \rho \text{ entangled.}$$

Notes:

- Key point: Need some way to prove that $tr(W\rho) \geq 0 \quad \forall \rho \in S$!
- Any given witness will only detect <u>some</u> entangled states.
- S is a convex set $\implies$ S is fully specified by all its tangent planes $\implies$ there exists a witness for any entangled states.

- Witnesses are <u>linear operators</u> $\Rightarrow$ they can be experimentally measured and can (and are) thus being used to identify entanglement in <u>experiments</u>.

Example:

$$W = \mathbb{F} := \sum_{i,j=1}^{d} |i,j\rangle\langle j,i| \quad \text{(the "flip" or swap)}$$

Is it a witness?

Let $\rho_{sep} = \sum p_i \rho_i^A \otimes \rho_i^B$:

$$tr(W\rho_{sep}) = \sum p_i \, tr\left(\mathbb{F}\left(\rho_i^A \otimes \rho_i^B\right)\right) = \ldots$$

Use:

(1) $\underline{tr[\mathbb{F} \cdot (A \otimes B)]} = \sum_{ij} tr\left[|ij\rangle\langle ji| \, A \otimes B\right]$

$$= \sum_{ij} \langle j,i| (A\otimes B) |i,j\rangle = \sum_{ij} \langle j|A|i\rangle\langle i|B|j\rangle$$

$$= \underline{tr(AB)} \quad \text{(the "magic formula")},$$

(2) $P, Q \geq 0 \Longrightarrow \mathrm{tr}(PQ) \geq 0$

Proof: $P = \sum p_i |\phi_i\rangle\langle\phi_i|$, $p_i \geq 0$.

$$\mathrm{tr}(PQ) = \sum_i \mathrm{tr}(p_i |\phi_i\rangle\langle\phi_i| Q)$$

$$= \sum_i p_i \underbrace{\langle\phi_i|Q|\phi_i\rangle}_{\geq 0} \geq 0.$$

$$\ldots = \sum p_i \underbrace{\mathrm{tr}(\rho_i^A \rho_i^B)}_{\geq 0} \geq 0.$$

$\Longrightarrow W$ is entanglement witness!

Which ent. states does W detect? — Those with dominant anti-symmetric component!

E.g. $d=2$, $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$:

$$\mathbb{F}|\psi^-\rangle = -|\psi^-\rangle \Longrightarrow \langle\psi^-|\mathbb{F}|\psi^-\rangle = -1 < 0\,!$$

while for e.g. $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$,

$$\mathbb{F}|\phi^+\rangle = |\phi^+\rangle \longrightarrow \langle\phi^+|\mathbb{F}|\phi^+\rangle = +1 > 0 \text{ ↯}$$

What about <u>mixed states</u>?

E.g. for $d=2$:

$$\rho = \lambda |\psi^-\rangle\langle\psi^-| - (1-\lambda)\frac{I}{4} \quad ; \quad \lambda \in [-\tfrac{1}{3}, 1]$$

"<u>Werner state</u>"

$$\mathrm{tr}(\mathbb{F}\rho) = \lambda \langle\psi^-|\mathbb{F}|\psi^-\rangle + \frac{(1-\lambda)}{4} \underbrace{\mathrm{tr}(I \cdot \mathbb{F})}_{= d = 2}$$

$$= -\lambda + \frac{1-\lambda}{2} = \frac{1}{2}(1-3\lambda)$$

$\rightarrow \rho$ entangled for $\lambda \geq 1/3$.

<u>Definition</u>: A witness W is called <u>optimal</u> if there exist ρ separable such that $\mathrm{tr}[W\rho]=0$.

(i.e., it touches the convex set & cannot be moved closer; otherwise, we could move it in parallel and get a strictly better witness.)

Is $W = \mathbb{F}$ optimal?

Yes, e.g. $\rho = |0\rangle\langle 0| \otimes |1\rangle\langle 1|$

$$\Rightarrow \operatorname{tr}(\rho \mathbb{F}) = 0.$$

Other witnesses: E.g. $W = I - d|\Omega\rangle\langle\Omega|$,

$$|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i,i\rangle \longrightarrow \text{Homework!}$$

c) Positive maps and the PPT criterion

Reminder: A superoperator $\Lambda : B(\mathcal{H}) \longrightarrow B(\mathcal{H})$ is called positive if $\rho \geq 0 \Rightarrow \Lambda(\rho) \geq 0$.

Usually – i.e. for physical maps – we want Λ in addition to be completely positive, i.e.

$$\rho_{AB} \geq 0 \Rightarrow (\Lambda_A \otimes I_B)(\rho_{AB}) \geq 0.$$

But now we will be interested in positive but not completely positive maps!

Why?

Consider $\rho_{sep} = \sum p_i \rho_i^A \otimes \rho_i^B$. Then,

$$(\Lambda \otimes I)(\rho_{sep}) = \sum p_i \underbrace{\Lambda(\rho_i^A)}_{=: \tilde{\rho}_i^A \geq 0 \; (\Lambda \text{ positive!})} \otimes \rho_i^B$$

$$= \sum \underbrace{p_i}_{\geq 0} \underbrace{\tilde{\rho}_i^A}_{\geq 0} \otimes \underbrace{\rho_i^B}_{\geq 0} \geq 0$$

and thus:

Theorem: Let $\Lambda : B(\mathcal{H}) \to B(\mathcal{H})$ be a positive map.

Then, $(\Lambda \otimes I)(\rho_{AB}) \ngeq 0 \implies \rho_{AB}$ entangled.

not positive semi-definite, i.e. has negative eigenvalues.

Most important example:

$$\Lambda(\rho) = \rho^T \quad (\text{the } \underline{\text{transpose map}})$$

$$(\Lambda \otimes I)(\rho) =: \rho^{T_A}$$ "partial transpose"

(cf. also section II.5)

E.g.: $|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i,i\rangle$

$$\Rightarrow (|\Omega\rangle\langle\Omega|)^{T_A} = \frac{1}{d} \left(\sum |i,i\rangle\langle j,j| \right)^{T_A}$$

$$= \frac{1}{d} \underbrace{\sum |j,i\rangle\langle i,j|}_{= \mathbb{F}!}$$

Not positive – antisymm. states have neg. eigenvalue,

i.e.: $|A\rangle = \frac{1}{\sqrt{2}} \left(|ij\rangle - |ji\rangle \right)$, $i \neq j$:

$$\langle A | (|\Omega\rangle\langle\Omega|)^{T_A} | A \rangle = \frac{1}{d} (-1) = -\frac{1}{d}.$$

Mixed states:

E.g. $\rho = \lambda |\Omega\rangle\langle\Omega| + (1-\lambda) \frac{I}{d^2}$, $\frac{-1}{d^2-1} \leq \lambda \leq 1$:

"isotropic state"

For instance for $d=2$:

$$\rho = \lambda \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} + (1-\lambda) \begin{pmatrix} \frac{1}{4} & & & \\ & \frac{1}{4} & & \\ & & \frac{1}{4} & \\ & & & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1+\lambda}{4} & & & \frac{\lambda}{2} \\ & \frac{1-\lambda}{4} & & \\ & & \frac{1-\lambda}{4} & \\ \frac{\lambda}{2} & & & \frac{1+\lambda}{4} \end{pmatrix}$$

T_A

$$\Longrightarrow \rho^{T_A} = \begin{pmatrix} \frac{1+\lambda}{4} & & & \\ & \frac{1-\lambda}{4} & \frac{\lambda}{2} & \\ & \frac{\lambda}{2} & \frac{1-\lambda}{4} & \\ & & & \frac{1+\lambda}{4} \end{pmatrix}$$

positive iff $\left(\frac{1-\lambda}{4}\right)^2 \geq \left(\frac{\lambda}{2}\right)^2$ & $\frac{1\pm\lambda}{4} \geq 0$

$\Longleftrightarrow -1 \leq \lambda \leq 1/3$

$\Longrightarrow \rho^{T_A} \geq 0$ iff $\lambda \in \left[\frac{-1}{d^2-1} ; \frac{1}{3}\right]$

$\Longrightarrow$ For $\lambda \in \left(\frac{1}{3}; 1\right]$, ρ is entangled

Corollary (PPT criteria):

$\rho^{T_A} \ngeq 0 \Longrightarrow \rho$ entangled.

This is called the PPT (positive partial transpose) criterion, or also NPT criterion (not positive...).

Note: PPT criterion – and generally positive maps – are invariant under local unitaries on B $\longrightarrow$ PPT detects all maximally ent. states (and in fact all states $|\chi\rangle$ w/ full Schmidt rank, as $|\chi\rangle = (I_A \otimes M_B)|\Omega\rangle$, M_B invertible).

That is: Positive maps are stronger than witnesses (will formalize this later) – but they cannot be measured.

In fact:

Theorem: The PPT criterion detects all entangled states in $d_A \times d_B = 2\times 2$ and 3×2 dimensions,

$$\rho \text{ entangled} \iff \rho^{T_A} \not\geq 0$$

(Not proven here.)

Counterexamples exist in $d_A \times d_B = 3\times 3$ and 2×4, i.e. states ρ with $\rho^{T_A} \geq 0$ which are entangled.

("PPT bound entangled states", cf. later.)

Other example of a positive but not CP map:

$$\Lambda(\rho) = \mathrm{tr}(\rho)\, I - \rho$$

$$(\Lambda \otimes I)(\rho_{AB}) = (I \otimes \mathrm{tr}_A \rho_{AB}) - \rho_{AB} .$$

$$I \otimes \mathrm{tr}_A \rho_{AB} - \rho_{AB} \not\geq 0 \implies \rho \text{ entangled.}$$

The "reduction criterion" for entanglement:

$$\underline{I \otimes \mathrm{tr}_A \rho \not\geq \rho} \qquad (\rightarrow \text{Homework}).$$

d) Relation between witnesses and positive maps:

For each witness W, there is a positive map which detects all entangled states the witness detects – and in fact more.

Connection can be understood via Choi-Jamiołkowski isomorphism.

Reminder (Choi-Jam. isomorphism):

CP map $\mathcal{E}$ on A $\qquad$ state σ on AB $(d_A = d_B)$

$$\mathcal{E} \longmapsto \sigma = (\mathcal{E} \otimes I)(|\Omega\rangle\langle\Omega|)$$

$$\mathcal{E}(\rho) = d\, \mathrm{tr}_B\left(\sigma\left(I \otimes \rho^T\right)\right) \longleftarrow\!\!\mapsfromchar\; \sigma$$

Isomorphism $\mathcal{E} \rightleftarrows \sigma$ applies also outside of CP maps and $\sigma \geq 0$.

Idea: Interpret W^T as the "Choi state" of a map Λ. (Since $W \not\geq 0$ – otherwise $\mathrm{tr}(W\rho) \geq 0\ \forall \rho$! – Λ is not CP.)

i.e.: $$\Lambda(X) := d\, \mathrm{tr}_B \left(W^T (I \otimes X^T) \right)$$
$$= d\, \mathrm{tr}_B \left(W (I \otimes X) \right)^T .$$

Then for $\rho \geq 0$:

$$\langle \phi | \Lambda(\rho)^T | \phi \rangle = d \langle \phi | \mathrm{tr}_B \left(W (I \otimes \rho) \right) | \phi \rangle$$

$$= d\, \mathrm{tr} \left[W \left(\underbrace{|\phi\rangle\langle\phi| \otimes \rho}_{\text{sep. state!}} \right) \right] \geq 0$$

W witness!

$\Longrightarrow$ Λ is a positive map!

and

$$\mathrm{tr}\left[W (A \otimes B) \right] = \mathrm{tr}_A \left[\mathrm{tr}_B \left(W (I \otimes B) \right) \cdot A \right]$$

$$= \frac{1}{d} \mathrm{tr}_A \left[\Lambda(B)^T A \right] = \frac{1}{d} \sum_{ij} \left[\Lambda(B)^T \right]_{ij} A_{ji}$$

$$= \langle \Omega | A \otimes \Lambda(B) | \Omega \rangle$$

$\overset{\text{linearity}}{\Longrightarrow}$ $$\mathrm{tr}(W \rho) = \langle \Omega | (I \otimes \Lambda)(\rho) | \Omega \rangle$$

<u>That is:</u>

$$\mathrm{tr}(W\rho) < 0 \implies (I \otimes \Lambda)(\rho) \not\geq 0$$

$\implies$ Λ detects all states which W detects.

<u>Conversely:</u> Witness W amounts to the map Λ, but checking for a negative eigenvalue only along $|\Omega\rangle$: <u>weaker</u> criterion!

<u>Corollary:</u> A state is separable if and only if

$(\Lambda \otimes I)(\rho) \geq 0$ $\forall$ positive maps Λ

(since sep. states $\leftrightarrow$ witnesses $\leftrightarrow$ positive maps).

<u>Example:</u> $W = \mathbb{F}$.

$$\Lambda(X) = d\,\mathrm{tr}_B\left(\mathbb{F}\left(I \otimes X^T\right)\right) = d \cdot X^T \cdot I = d X^T$$

"magic formula" for partial trace!

$\implies$ $W = \mathbb{F}$ corresponds to PPT criterion!

(But: PPT detect all Bell states, $\mathbb{F}$ only the antisym. state)

e) Quantification of mixed state entanglement

How can we quantify entanglement of mixed states?

i) Entanglement needed to create the state, e.g.

minimal amount of $E(|\psi\rangle) = S(\mathrm{tr}_B |\psi\rangle\langle\psi|)$ needed:

$$E_F(\rho) := \min_{\substack{\{p_i, |\psi_i\rangle\} \\ \text{s.th. } \sum p_i |\psi_i\rangle\langle\psi_i| = \rho}} \sum p_i E(|\psi_i\rangle)$$

"entanglement of formation"

or an asymptotic version (cost per copy)

$$E_C(\rho) := \lim_{N\to\infty} \frac{1}{N} E_F(\rho^{\otimes N})$$

"entanglement cost"

Already E_F very hard to compute – need to minimize concave function $S(\rho)$ over convex set of decompositions $\rho = \sum p_i \rho_i$ – i.e. minimum obtained at boundary (hard!)

Also, $E_F(\rho \otimes \sigma) \neq E_F(\rho) + E_F(\sigma)$

(additivity of E_F – counterex. exist (Hastings))

<u>But</u>: <u>Analytic formula</u> for <u>$d_A \times d_B = 2 \times 2$</u> for E_F exists (Wootters), based on so-called "<u>concurrence</u>".

<u>ii) Extractable entanglement:</u>

"<u>Distillable entanglement</u>" $E_D(\rho)$

$E_D(\rho)$ = max. asympt. rate $R = \frac{M}{N}$ achievable for LOCC-protocol $\mathcal{E}_N$: $\| \mathcal{E}_N(\rho^{\otimes N}) - |\phi^+\rangle\langle\phi^+|^M \| \to 0$ as $N, M \to \infty$.

suitable dist. meas., typ. trace norm.

<u>Even harder</u> to compute:

Asymptotic ($N, M \to \infty$) and any # of LOCC rounds.

(Versions w/ restricted LOCC rounds exist, e.g. one-way dist. entanglement, ...)

<u>Observation:</u> $E_F(\rho) \geq E_C(\rho) \geq E_D(\rho)$

Generally, $E_C(\rho) \gneq E_D(\rho)$: for most states, process is <u>not reversible</u>.

$\Longrightarrow$ no unique measure as for pure states!

<u>Example:</u> ρ PPT, i.e. $\rho^{T_A} \geq 0$.

LOCC preserves PPT $\Longrightarrow$ PPT states are <u>undistillable</u>, $E_D(\rho) = 0$.

But there exist PPT states with $E_C(\rho) > 0$:

"PPT bound entangled states"

(<u>Note:</u> The converse problem –

does $\rho^{T_A} \ngeq 0 \Longrightarrow E_D(\rho) > 0$ hold?

– is a big open problem – the existence of NPT bound entangled states.)

Problem: These might be natural ent. measures, but they are essentially impossible to compute.

$\Rightarrow$ Computable ent. measure desirable!

Wishlist for a good ent. measure:

- LOCC-monotone: Cannot be increased by LOCC (probably most relevant!)
- $E(\rho)=0$ for sep. states ρ (and only there?)
- additive: $E(\rho\otimes\sigma) = E(\rho) + E(\sigma)$
- continuous: $\underset{\text{typ. trace norm}}{\rho \approx \sigma} \Rightarrow E(\rho) \approx E(\sigma)$
- $E_D \leq E \leq E_C$.
- coincides with $E(|\psi\rangle) = S(\mathrm{tr}_B |\psi\rangle\langle\psi|)$ on pure states.
- computable?

(Almost) impossible to get all – LOCC monotonicity is probably the most relevant one.

Negativity – a computable entanglement measure

Found previously:

ρ^{T_A} has neg. eigenvalues $\Rightarrow$ ρ entangled.

Use negative eigenvalues to measure entanglement:

"**Negativity**"

$$\mathcal{N}(\rho) = \frac{1}{2}\Big(\underbrace{\sum_i \big|\overbrace{\lambda_i(\rho^{T_A})}^{\text{eigenvalues}}\big|}_{=: \|\rho^{T_A}\|_1 \,:\, \text{trace norm}} - 1 \Big)$$

$$= \frac{1}{2}\left(\|\rho^{T_A}\|_1 - 1 \right)$$

$$= - \sum_{\text{neg. eigenvalues}} \lambda_i(\rho^{T_A})$$

holds since $\sum \lambda_i(\rho^{T_A}) = tr(\rho^{T_A})$
$= tr(\rho) = 1$

or "**Log-negativity**"

$$E_N(\rho) = \log_2 \|\rho^{T_A}\|_1$$

Properties:

Negativity $\mathcal{N}$:

- LOCC - monotone
- not additive
- $\mathcal{N}(\rho) = 0$ for ρ separable,

  but also $\mathcal{N}(\rho)$ on PPT ent. states!

- $\neq E(|\psi\rangle)$ for pure states
- continuous

Log-negativity E_N:

- not an LOCC monotone (!)
- additive
- $E_N(\rho) = 0$ for ρ sep, but also on PPT ent. st.
- $\neq E(|\psi\rangle)$ for pure states
- continuous