

V. Quantum Error Correction

1. Introduction

a) Setting & Problem

- Coupling to environment induces errors (i.e., uncontrolled behavior).
- Classical computers: information stored in "macroscopic" properties \rightarrow errors unlikely.
- Quantum computers:
 - need qubits = "single" quantum systems, and must store general superposition, not just $|0\rangle$ and $|1\rangle$
 \rightarrow fragile!
 - should be well isolated to protect qubits, but also need coupling to "environment" (experimental apparatus) to control the computation (gates, measurements).

Q: Can we protect quantum information from noise?

Classical error correction:

Copy information, e.g. encode 1 bit in 3 bits:

$$0 \mapsto \hat{0} := 000$$

$$1 \mapsto \hat{1} := 111$$

"encoding"

Error model: Bit flip w/ some (small) probability p

(independently on all bits):

\Rightarrow typically 0 or 1 bits flipped.

Error correction ("decoding") by majority vote:

$$000, 001, 010, 100 \mapsto 000$$

$$111, 110, 101, 011 \mapsto 111$$

Probability for a "logical error" (i.e. on encoded bit):

$$P_{\text{error}} = \text{prob}(\geq 2 \text{ flips}) = p^3 + 3p^2(1-p)$$

$$= 3p^2 - 2p^3 < p \quad \text{for } p < 1/2.$$

\curvearrowright error quadratically suppressed!

\Rightarrow effective error probability decreased.

Can be improved by:

- using more bits: $0 \mapsto 00\dots 0$, $1 \mapsto 11\dots 1$
- using ("concatenating") codes
- using smarter codes (i.e. encode several bits at once)

Quantum Error Correction:

Several potential problems when trying to generalize classical error correction codes:

- cannot copy qubits
- even if we could: what would be the "majority vote"?
- different types of errors exist,
 e.g. X (bit flip)
 or Z ("phase flip")
- errors can be continuous: there is an infinity of errors!
- measuring qubits destroys quantum information!

6) The 3-qubit bit flip code

Copy qubits in computational basis:

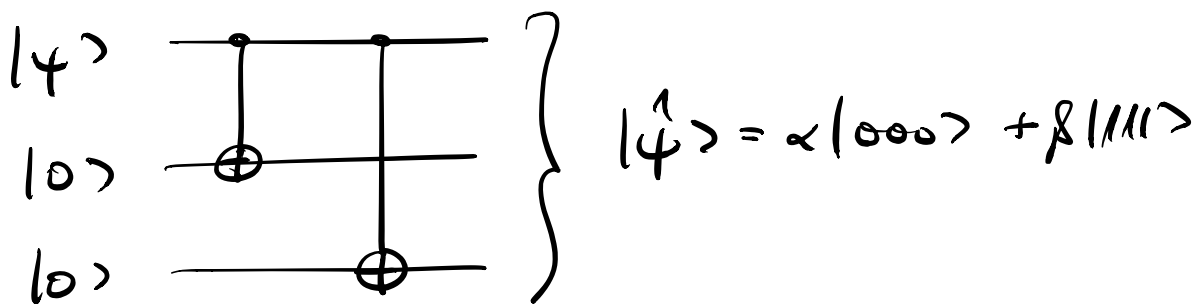
$$|0\rangle \mapsto |\hat{0}\rangle = |000\rangle$$

$$|1\rangle \mapsto |\hat{1}\rangle = |111\rangle$$

i.e., the encoding is a linear map

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \xrightarrow{\text{encoding}} \alpha|000\rangle + \beta|111\rangle$$

Possible encoding circuit:



Now consider bit flip error on qubit i :

$$|\hat{\psi}\rangle \xrightarrow{\text{error}} X_i |\hat{\psi}\rangle$$

Can we correct for one bit flip error on an unknown qubit i ?

Problem: Measuring the qubit's in comp. basis reveals i , but also destroys superposition!

⇒ Need a measurement which only returns information about position i of error - indep. of encoded state $|\psi\rangle$!

Define "syndrome measurement" with outcomes 0, 1, 2, 3, and projectors:

0 = "no flip": $P_0 = |000\rangle\langle 000| + |111\rangle\langle 111|$

1 = "1st qubit flipped": $P_1 = |100\rangle\langle 100| + |011\rangle\langle 011|$

2 = "2nd qubit flipped": $P_2 = |010\rangle\langle 010| + |101\rangle\langle 101|$

3 = "3rd qubit flipped": $P_3 = |001\rangle\langle 001| + |110\rangle\langle 110|$

(This defines a complete measurement, as $\sum P_i = I$)

The outcome is called the "error syndrome".

Measurement of $\{P_i\}$ reveals only 2 bits of info.

⇒ one qubit of information untouched!

By direct inspection: The information obtained is the location of the bit flip, e.g.

$$\alpha|000\rangle + \beta|111\rangle \xrightarrow[\text{on qubit 2}]{\text{bit flip}} \alpha|010\rangle + \beta|101\rangle$$

\Rightarrow measurement always returns P_2 ,

with post-measurement state

$$\alpha|010\rangle + \beta|101\rangle \xrightarrow[\text{flip qubit 2}]{\text{recovery:}} \alpha|000\rangle + \beta|111\rangle !$$

\Rightarrow Bit flip corrected!

Works for any single bit flip in unknown location and no flip, and for all states $|\psi\rangle$

\Rightarrow suppression of error $p \rightsquigarrow 3p^2 - 2p^3$, as classically.

By linearity, this also works for part of a larger entangled state:

$$\alpha|0\rangle|a\rangle + \beta|1\rangle|b\rangle \xrightarrow{\text{encode}} \alpha|000\rangle|a\rangle + \beta|111\rangle|b\rangle$$

$$\xrightarrow[\text{X}_1]{\text{error:}} \alpha|100\rangle|a\rangle + \beta|011\rangle|b\rangle \xrightarrow[\text{Correct: X}_1]{\text{meas.: } P_1} \alpha|000\rangle|a\rangle + \beta|111\rangle|b\rangle$$

Measurement of error syndrome $\{P_\alpha\}$ collapses
continuous error into one of the 4 correctable

discrete errors:

- measurement "digitalizes" error
- sufficient to study discrete (detectable) errors (will be formalized later)

A different perspective on syndrome measurement & correction (the "stabilizer formalism" - more later):

$|000\rangle, |111\rangle$: +1 eigenstates of $Z_1 Z_2$ and $Z_2 Z_3$
("stabilizers")

Measure $Z_1 Z_2$ and $Z_2 Z_3$:

compare qubits 1&2 and 2&3

- (+1, +1) : no error
- (-1, +1) : qubit 1 flipped
- (+1, -1) : qubit 3 flipped
- (-1, -1) : qubit 2 flipped

Now formally:

encoded state $|\hat{\psi}\rangle = \alpha|000\rangle + \beta|111\rangle$:

$$\Rightarrow z_1 z_2 |\hat{\psi}\rangle = |\hat{\psi}\rangle, \quad z_2 z_3 |\hat{\psi}\rangle = |\hat{\psi}\rangle$$

But flip error, e.g. X_1 :

X_1 anti-commutes with $z_1 z_2$

$$\begin{aligned} \Rightarrow \langle \hat{\psi} | X_1 z_1 z_2 X_1 | \hat{\psi} \rangle &= - \langle \psi | z_1 z_2 | \psi \rangle \\ &= -1 \end{aligned}$$

Thus:

Outcome -1 for $z_1 z_2 \iff$ an error
which anti-commutes with $z_1 z_2$ has
occurred.

The correction operation must satisfy the same
anti-commutation relations (and some further
properties) \rightarrow lets!

Have focused on X errors.

But what about Z errors?

$$\alpha|000\rangle + \beta|111\rangle \xrightarrow[\text{on qubit 1}]{Z \text{ error}} \alpha|000\rangle - \beta|111\rangle$$

This is still a state in the code space

(i.e., a valid encoded state $|\psi\rangle$)

\Rightarrow error not detectable, but it has changed

$|\psi\rangle$. After decoding, the error acts as

$$\alpha|0\rangle + \beta|1\rangle \longrightarrow \alpha|0\rangle - \beta|1\rangle,$$

i.e. as a logical Z operation,

“logical operation” =
operation on encoded qubit.

\Rightarrow 3-qubit bit flip code cannot protect
against single “phase flip error” Z.

Stabilizer picture:

Error Z_i commutes with stabilizers Z_1Z_2 & Z_2Z_3
 \implies it cannot be detected.

But: Z_i cannot be expressed as a product of the stabilizers Z_1Z_2 & $Z_2Z_3 \implies$ Logical error!

c) The 3-qubit phase-flip code

Can we correct against Z errors?

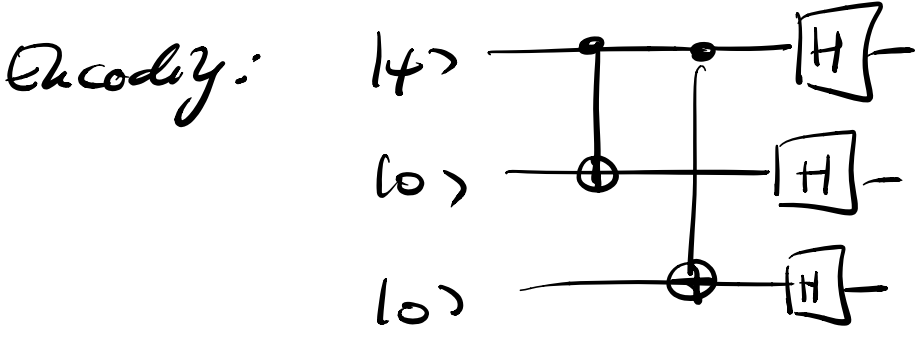
$$Z|+\rangle = |- \rangle, \quad Z|- \rangle = |+\rangle$$

$\implies Z$ error $\hat{=}$ bit flip error in $| \pm \rangle$ -basis.

Use encoding $\alpha|0\rangle + \beta|1\rangle \longmapsto \alpha|\hat{0}\rangle + \beta|\hat{1}\rangle,$

with $|\hat{0}\rangle := |+++ \rangle, \quad |\hat{1}\rangle := |-- \rangle.$

Will protect against single Z errors!



Syndrome measurement:

$$\tilde{P}_\alpha := H^{\otimes 3} P_\alpha H^{\otimes 3}$$

(or via stabilizers X_1, X_2 & $X_2 X_3$).

Recovery operation:

$$H X_i H = Z_i$$

(anti-com. with stabilizers).

Problem:

Now, there is no protection against bit flip errors X_i :

— and X_i acts as a logical X operator!