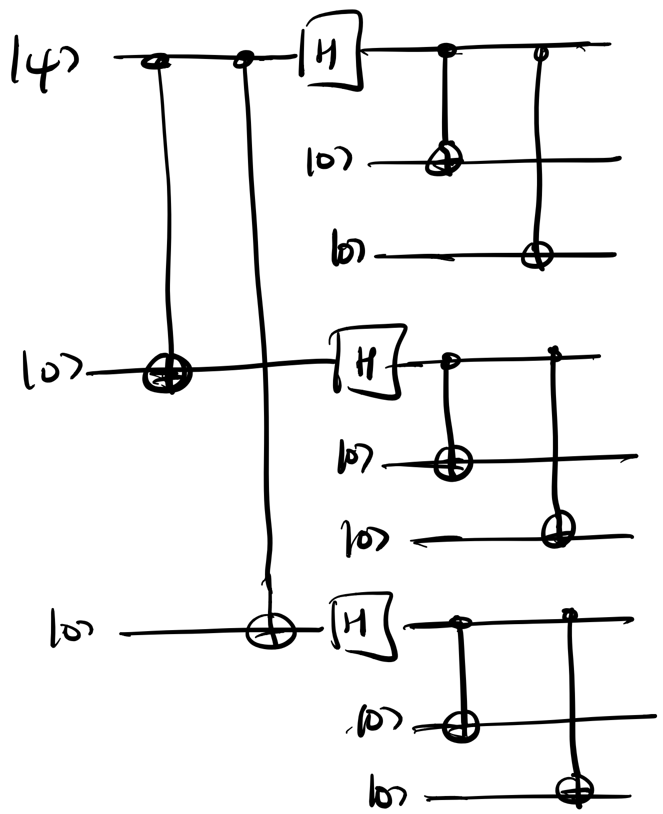
2. The 9-qubit Shor code

<u>Solution:</u> Concatenate (= nest) 3-qubit bit flip code and 3-qubit phase flip code:

$$|0\rangle \longmapsto |+\rangle|+\rangle|+\rangle \longmapsto \frac{(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)}{2\sqrt{2}}$$

$$|1\rangle \longmapsto |-\rangle|-\rangle|-\rangle \longmapsto \frac{(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)}{2\sqrt{2}}$$

Encoding circuit:



9-qubit Shor code

Shor code protects against arbitrary single-qubit errors!

Sufficient to focus on X, Z, and $Y \propto XZ$ errors:

Any general error $E = \alpha I + \sum \beta_i \sigma_i$ will collapse to one of these (if done right).

— More on this later! —

Intuitively:

(i) Errors X_i are corrected on "inner" layer.

(ii) Z_i error =

= logical error on qubit encoded in inner layer

= Z error on outer layer in one position

$\Longrightarrow$ correctable!

(iii) $Y_i \propto X_i Z_i$:

Correct X_i on inner layer.

Then as in (ii): only Z error left!

More formally: Stabilizers

Code is +1 eigenstate of

$Z_1 Z_2, \quad Z_2 Z_3$ ← 1st inner layer

$Z_4 Z_5, \quad Z_5 Z_6$

$Z_7 Z_8, \quad Z_8 Z_9$

and of

X on intermediate qubits

$X_{(123)} X_{(456)}$

$\Downarrow \quad \alpha|000\rangle + \beta|111\rangle$

$X_{(456)} X_{(789)},$

$X_{(123)} = X_1 X_2 X_3$

$\Longrightarrow$

$X_1 X_2 X_3 X_4 X_5 X_6$

$X_4 X_5 X_6 X_7 X_8 X_9$

These are 8 commuting operators: Measuring gives 8 bits of information ⟹ 1 qubit untouched!

Analysis of errors:

Bit flip error X_i:

e.g. X_1 anti-comm. w/ $Z_1 Z_2$

or X_2 anti-com. w/ $Z_1 Z_2$ & $Z_2 Z_3$

$\Rightarrow$ meas. of all 6 $Z_k Z_l$ reveals position of X_i

$\Rightarrow$ can be corrected!

Phase flip error Z_i:

E.g.: Z_1: anti-com. w/ $X_1 X_2 X_3 X_4 X_5 X_6$

But: same holds for Z_2 or Z_3!

Yet: Z_1, Z_2, and Z_3 act identically on encoded state $|\hat{\psi}\rangle$ – can be seen by inspection, or since

$$Z_2 |\hat{\psi}\rangle = Z_1 \underbrace{(Z_2 Z_1) |\hat{\psi}\rangle}_{= |\hat{\psi}\rangle \text{ (stabilizer!)}} = Z_1 |\hat{\psi}\rangle!$$

(The 9-qubit code is a <u>degenerate code</u>: different errors have the same syndrome!)

<u>Y errors Y_i</u>:

E.g. $Y_2 \propto Z_2 X_2$

anti-comm. w/ $Z_1 Z_2$

$Z_2 Z_3$

$X_1 X_2 X_3 X_4 X_5 X_6$

$\to$ correction e.g. via $Z_2 X_2$, or $Z_1 X_2$, ...

All single-qubit errors can be corrected!

What if errors occur on <u>more than one qubit</u>?

Some – but not all! – can be corrected:

e.g. $X_1 X_4$: correctable.

$Z_1 Z_2$: trivial = no error

but: $X_1 X_2$: breaks inner code ↯

$Z_1 Z_4$: breaks outer code ↯