3. The Quantum Error Correction Conditions

Definition: Given $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$, a Quantum Error Correction Code (QECC) on $\mathcal{H}$ is a sub-space $\mathcal{C} \subset \mathcal{H}$ (the code space, with $|\psi\rangle \in \mathcal{C}$ codewords).

We denote by $|\hat{\imath}\rangle$ an (arbitrary, but fixed) basis of $\mathcal{C}$.

Definition: A noise model on $\mathcal{H}$ is a CPTP map

$$\mathcal{E}(\rho) = \sum E_\alpha \rho E_\alpha^\dagger \;; \quad \sum E_\alpha^\dagger E_\alpha = I$$

(i.e., error E_α occurs w/ prob. $\mathrm{tr}(E_\alpha^\dagger E_\alpha \rho)$,

e.g. $E_\alpha \propto$ single-qubit Paulis.)

Definition: We say that a QECC $\mathcal{C}$ can correct for an error $\mathcal{E}$ if there exists a recovery map $\mathcal{R}$, i.e. a CP map $\mathcal{R}$ such that

$$\mathcal{R}(\mathcal{E}(\rho)) = \rho \qquad \forall \rho = |\hat{\psi}\rangle\langle\hat{\psi}|,\ |\hat{\psi}\rangle \in \mathcal{C}$$

(Note: This implies that $\mathcal{R}$ is trace-preserving on states supported on the image of $\mathcal{C}$ under $\mathcal{E}$, i.e., on states obtained by noise from a code state.)

Theorem (Quantum Error Correction Condition):

Given $\mathcal{C}$ and $\mathcal{E}(\cdot) = \sum E_\alpha \cdot E_\alpha^\dagger$, there exists a recovery $\mathcal{R}$ (i.e. $\mathcal{C}$ corrects for $\mathcal{E}$) if and only if

$$\boxed{\langle \hat{i} | E_\alpha^\dagger E_\beta | \hat{j} \rangle = c_{\alpha\beta}\, \delta_{ij}} \qquad (*)$$

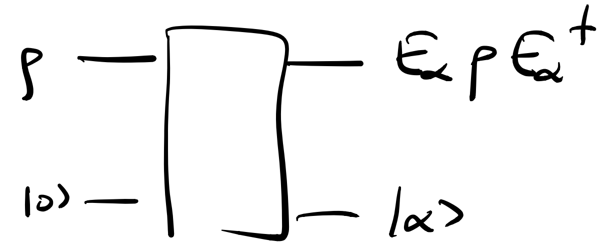
for some ONB $\{|\hat{i}\rangle\}$ ($\langle \hat{i} | \hat{j} \rangle = \delta_{ij}$) of $\mathcal{C}$.

Intuition:

① Orthogonal states remain orthogonal (R cannot make states more orthogonal!)

② Environment learns nothing about state:

Stinespring:



$$\text{prob}(\alpha) = \langle \hat{i} | E_\alpha^\dagger E_\alpha | \hat{i} \rangle = c_{\alpha\alpha} \quad \text{indep. of } i;$$

$$\text{prob}(\alpha) = \Big(\sum \bar{a}_i \langle \hat{i} | \Big) E_\alpha^\dagger E_\alpha \Big(\sum a_j | \hat{j} \rangle \Big)$$

$$= \sum |a_i|^2 c_{\alpha\alpha} \quad \text{indep. of } i.$$

Proof:

'existence of $R \Longrightarrow \circledast$':

Lemma: $\sum_\tau K_\tau |\psi\rangle\langle\psi| K_\tau^\dagger = |\psi\rangle\langle\psi| \quad \forall |\psi\rangle \in \mathcal{C}$

$\Longrightarrow K_\tau |\psi\rangle = a_\tau |\psi\rangle$

with a_τ indep. of $|\psi\rangle$.

Proof: $\sum_\tau K_\tau |\psi\rangle\langle\psi| K_\tau^\dagger = |\psi\rangle\langle\psi|$

Choose any $|\chi\rangle$ s.th. $\langle\chi|\psi\rangle = 0$

$$\Rightarrow \sum_\tau \underbrace{\langle\chi|K_\tau|\psi\rangle\langle\psi|K_\tau^\dagger|\chi\rangle}_{\geq 0} = \langle\chi|\psi\rangle\langle\psi|\chi\rangle = 0$$

$$\Rightarrow \langle\chi|K_\tau|\psi\rangle = 0 \quad \forall \tau$$

$$\Rightarrow K_\tau|\psi\rangle = a_\tau(|\psi\rangle)\,|\psi\rangle.$$

What if $a_\tau(|\psi\rangle)$ dep. on $|\psi\rangle$? Choose $|\psi_1\rangle, |\psi_2\rangle$ s.th. $a_\tau(|\psi_1\rangle) \neq a_\tau(|\psi_2\rangle)$. Then,

$$K_\tau(|\psi_1\rangle + |\psi_2\rangle) = a_\tau(|\psi_1\rangle)|\psi_1\rangle + a_\tau(|\psi_2\rangle)|\psi_2\rangle$$

$$\not\propto |\psi_1\rangle + |\psi_2\rangle \quad ↯$$

$$\Rightarrow a_\tau(|\psi\rangle) \equiv a_\tau$$

$$\Rightarrow K_\tau|\psi\rangle = a_\tau|\psi\rangle. \qquad \blacksquare$$

Let $\mathcal{R}(\cdot) = \sum R_\gamma \cdot R_\gamma^\dagger.$

Then: $\mathcal{R}(\mathcal{E}(|\psi\rangle\langle\psi|)) = |\psi\rangle\langle\psi| \quad \forall |\psi\rangle \in \mathcal{C}$

Lemma $\Longrightarrow$ $R_\gamma E_\alpha |\psi\rangle = a_{\gamma\alpha} |\psi\rangle \quad \forall |\psi\rangle \in \mathcal{C}$

ONB $|\hat{i}\rangle, |\hat{j}\rangle$:

$$\Longrightarrow \sum_\gamma \langle \hat{i} | E_\alpha^\dagger R_\gamma^\dagger R_\gamma E_\beta | \hat{j} \rangle = \sum_\gamma \bar{a}_{\gamma\alpha} a_{\gamma\beta} \langle \hat{i} | \hat{j} \rangle$$

$$=: c_{\alpha\beta} \, \delta_{ij}$$

$$\Longrightarrow \langle \hat{i} | E_\alpha^\dagger \underbrace{\Big(\sum_\gamma R_\gamma^\dagger R_\gamma \Big)}_{= I \text{ on image of } \mathcal{C} \text{ under } \mathcal{E}.} E_\beta | \hat{j} \rangle = c_{\alpha\beta} \, \delta_{ij}$$

$$\Longrightarrow \langle \hat{i} | E_\alpha^\dagger E_\beta | \hat{j} \rangle = c_{\alpha\beta} \, \delta_{ij} \qquad \blacksquare$$

"$\circledast \Longrightarrow$ existence of $\mathcal{R}$":

Construct explicit recovery channel $\mathcal{R}(\cdot) = \sum R_\gamma \cdot R_\gamma^\dagger$.

Step 1: Use gauge degree of freedom in E_α:

$$\mathcal{E}(\rho) = \sum E_\alpha \rho E_\alpha^\dagger = \sum F_\beta \rho F_\beta^\dagger$$

if & only if $F_\beta = \sum_\alpha V_{\beta\alpha} E_\alpha$, V isometry.

Choose V s.th. $\sum_{\alpha\beta} V_{\varepsilon\alpha}^* c_{\alpha\beta} V_{\tau\beta} = \lambda_\varepsilon \delta_{\varepsilon\tau}$ diagonal

$$\circledast \Longrightarrow \langle \hat{i} | F_\varepsilon^+ F_\tau | \hat{j} \rangle = \sum_{\alpha,\beta} \langle \hat{i} | \overline{V_{\varepsilon\alpha}} E_\alpha^+ E_\beta V_{\tau\beta} | \hat{j} \rangle$$

$$= \sum_{\alpha,\beta} \overline{V_{\varepsilon\alpha}} V_{\tau\beta} \langle \hat{i} | E_\alpha^+ E_\beta | \hat{j} \rangle$$

$$= \sum_{\alpha\beta} \overline{V_{\varepsilon\alpha}} V_{\tau\beta} \; c_{\alpha\beta} \; \delta_{ij}$$

$$= \lambda_\varepsilon \delta_{\varepsilon\tau} \delta_{ij}$$

$\Longrightarrow$ Different errors F_ε can be distinguished by a projective measurement!

Note that $\sum_\varepsilon \lambda_\varepsilon = \sum_\varepsilon \underbrace{\langle \hat{i} | F_\varepsilon^+ F_\varepsilon | \hat{i} \rangle}_{= \lambda_\varepsilon \text{ : prob. of error } \varepsilon} = \langle \hat{i} | I | \hat{i} \rangle = 1.$

Step 2 : Measure ε and undo error F_ε.

Want R_γ s.th. $R_\gamma F_\varepsilon | \hat{i} \rangle = \sqrt{\lambda_\varepsilon} \, \delta_{\gamma\varepsilon} | \hat{i} \rangle$!

prob. of error F_ε.

Choose $R_\gamma := \dfrac{1}{\sqrt{\lambda_\varepsilon}} \sum_j | \hat{j} \rangle \langle \hat{j} | F_\gamma^+ .$

If $\lambda_\varepsilon = 0$, then $R_\gamma = 0$ is a solution.

$$\Longrightarrow R_\gamma F_\varepsilon | \hat{i} \rangle = \frac{1}{\sqrt{\lambda_\varepsilon}} \sum_j | \hat{j} \rangle \underbrace{\langle \hat{j} | F_\gamma^+ F_\varepsilon | \hat{i} \rangle}_{= \lambda_\varepsilon \delta_{\gamma\varepsilon} \delta_{ij}} = \sqrt{\lambda_\varepsilon} \, \delta_{\gamma\varepsilon} | \hat{i} \rangle .$$

$$\Rightarrow R_\gamma F_\varepsilon |\hat{\psi}\rangle = \sqrt{\lambda_\varepsilon}\, \delta_{\gamma\varepsilon} |\hat{\psi}\rangle \quad \forall |\hat{\psi}\rangle \in \mathcal{C}$$

$$\Rightarrow Q(\mathcal{E}(|\hat{\psi}\rangle\langle\hat{\psi}|)) = \sum_{\gamma,\varepsilon} R_\gamma F_\varepsilon |\hat{\psi}\rangle\langle\hat{\psi}| F_\varepsilon^\dagger R_\gamma^\dagger$$

$$= \sum_\varepsilon \lambda_\varepsilon |\hat{\psi}\rangle\langle\hat{\psi}| = |\hat{\psi}\rangle\langle\hat{\psi}| \quad \forall |\hat{\psi}\rangle \in \mathcal{C} \quad \boxtimes$$

<u>Note</u>: For any single-qubit error, we have (by definition)

$$E_\alpha = \sum_{k,s} w_{\alpha,k,s}\, \sigma_s^k$$

k'th Pauli matrix on qubit s.

and $\langle\hat{i}|\sigma_s^k \sigma_r^l|\hat{j}\rangle \propto \delta_{ij} \Rightarrow \langle\hat{i}|E_\alpha^\dagger E_\beta|\hat{j}\rangle \propto \delta_{ij}$.

<u>Thus</u>: Error Correction Condition holds for Paulis

$\Rightarrow$ error correction condition holds for any single-qubit error!

<u>In particular</u>: A QECC which can correct for <u>depolarizing noise</u>

$$\mathcal{E}(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z)$$

on any qubit is also robust against <u>any</u> <u>single-qubit error!</u>

<u>Corollary:</u> To check for robustness against arbitrary single-qubit errors, it is sufficient to check the error model with

$$\{E_\alpha\} \propto \{I, X, Y, Z\}$$

E_α up to prefactors

The analogous result holds for k-qubit errors vs. k-qubit Paulis.

<u>Exercise suggestion:</u> Check q. error correction condition for 3-qubit & 9-qubit code!