4. Basic properties of QECCs

Focus on "binary codes":

encode k qubits in n $n > k$ qubits

Definition: The distance d of a QECC is the smallest number of Paulis $\{P_{i_k} \neq I\}_{k=1}^{d}$ s.t.

$$\langle \hat{i} | F | \hat{j} \rangle \neq \lambda \delta_{ij} \quad \text{for some } |\hat{i}\rangle, |\hat{j}\rangle \in \mathcal{C}, \; \langle \hat{i} | \hat{j} \rangle = \delta_{ij}.$$

where $F := P_{i_d} \otimes I \otimes \ldots \otimes P_{i_2} \otimes I \otimes \ldots$

(i.e.: The smallest # of sites where we have to apply a Pauli to change a code state into another.)

Notation: A binary code encoding k qubits in n qubits with distance d is denoted

$$[\![n, k, d]\!] \text{-code}$$

phys. qubits — logical qubits — distance

How many one-qubit errors can a distance-d code correct for?

Can focus on Pauli errors.

For E_α, E_β with $\leq t$ Paulis each:

$$\langle i | \underbrace{E_\alpha^\dagger E_\beta}_{\leq 2t \text{ Paulis}} | j \rangle \overset{?}{=} c_{\alpha\beta} \delta_{ij} \quad \forall E_\alpha, E_\beta$$

$$\Longleftrightarrow \quad 2t + 1 \leq d$$

Result: A distance-d code can correct t general single-qubit errors if & only if

$$\boxed{2t + 1 \leq d}$$

E.g. with a $d = 3$-code, we can correct any single-qubit error.

If the location of the error is known – that is, we additionally learn that a specific noise channel $\mathcal{E}_{\text{location}}(\cdot) = \sum \tilde{E}_\alpha \rho \tilde{E}_\alpha^\dagger$ has been applied:

$$\langle \hat{i} | \tilde{E}_\alpha^\dagger \tilde{E}_\beta | \hat{j} \rangle$$

Paulis on same location

$\Rightarrow \tilde{E}_\alpha^\dagger \tilde{E}_\beta$ has $\leq t$ Paulis

$\Rightarrow$ correctable for $\boxed{t+1 \leq d}$

Result: QECC can correct t errors on unknown locations $\Leftrightarrow$ QECC can correct $2t$ errors on known locations.

What are constraints on $[\![n,k,d]\!]$?

Definition: A code is called **non-degenerate** if different Pauli errors result in orthogonal

states, i.e. are distinguishable,

$$\langle \hat{j} | E_\alpha^\dagger E_\beta | \hat{i} \rangle \propto \delta_{\alpha\beta}$$

for all E_α s/ at most t ($2t+1 \leq d$) Paulis.

Theorem (Hamming bound):

For non-degenerate codes,

$$\sum_{j=0}^{t} 3^j \binom{n}{j} \leq 2^{n-k}, \quad 2t+1 = d.$$

Proof: n'a counting possibilities. (→ exercise!)

E.g.: For $k=1$, $t=1$ ($d=3$) — i.e. encodes 1 qudit, can correct for one error:

$$n \geq 5.$$

Could there be a degenerate $[[4,1,3]]$-code?

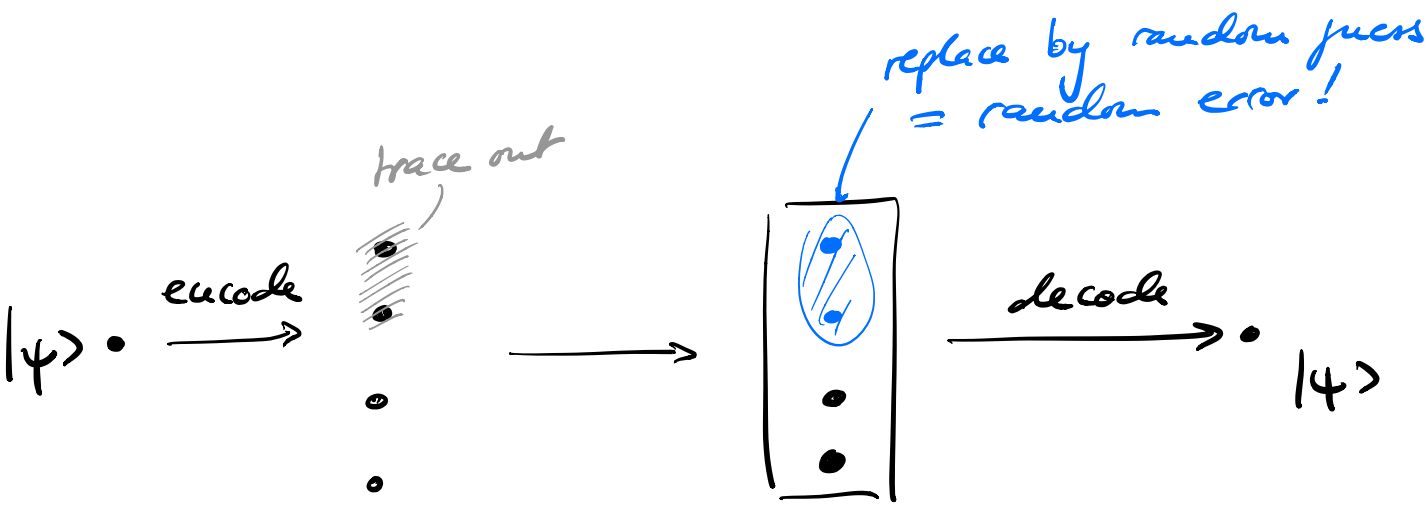
NO!

Proof:

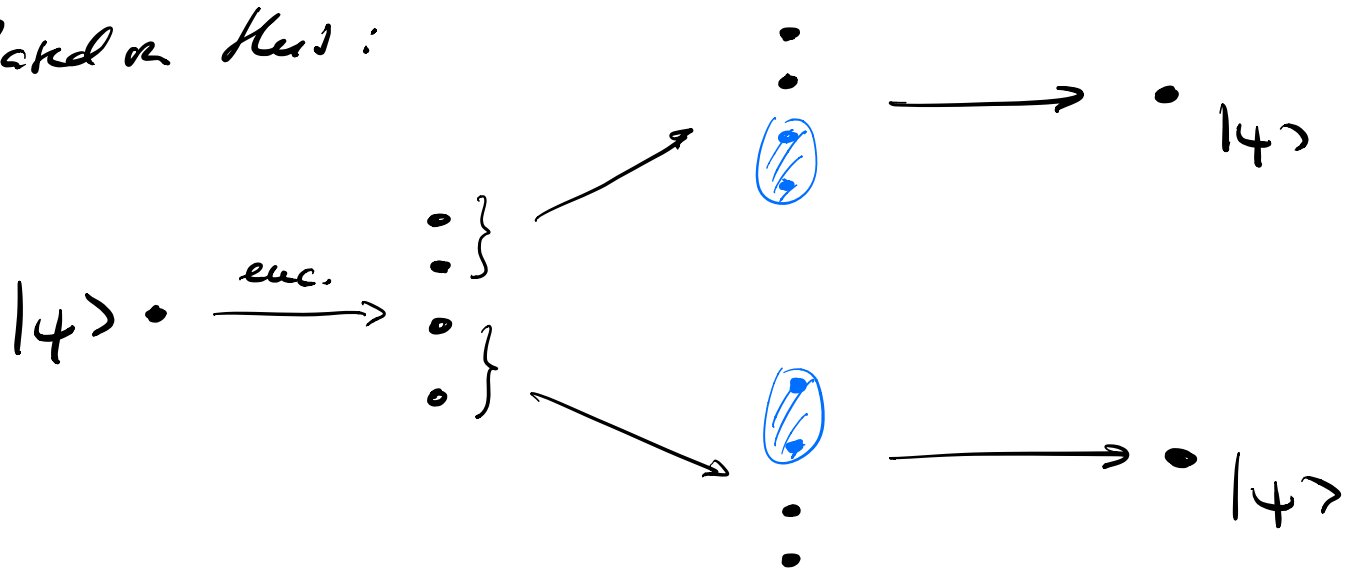
$d=3$: Can correct for unknown 1-qubit error

$\Rightarrow$ can correct for 2 errors in known location

Can use it to recover 2 lost qubits:



Back to Alice:



↳ have built a quantum closet

→ No $[\![4,1,3]\!]$ code can exist,

a $[\![5,1,3]\!]$ code would be optimal!