5. Stabilizer codes

Have seen, e.g. for 3-qubit / 9-qubit code:

code space = joint +1 eigenspace of Paulis

error & correction $\longleftrightarrow$ anti-comm. pattern

$\longrightarrow$ General framework?

a) Definition

Definition: The Pauli group $\mathcal{G} \equiv \mathcal{G}_n$ on n qubits is

$$\mathcal{G} := \{ i^\ell P_1 \otimes \ldots \otimes P_n \mid P_i = I, X, Y, Z \, ; \, \ell = 0, \ldots, 3 \}$$

Note: Any two $S_1, S_2 \in \mathcal{G}$ either commute or anti-commute.

Definition (Stabilizer group, stabilizer code)

A subgroup $\mathcal{S} \subset \mathcal{G}$ with $-I \notin \mathcal{S}$ is called a stabilizer group $\mathcal{S}$. Since $-I \notin \mathcal{S} \Rightarrow S_1, S_2 \in \mathcal{S}$ commute (else $S_1 S_2 S_1 S_2 = -I$); this also implies $S = \pm \otimes P_i \quad \forall S \in \mathcal{S}$.

The elements $S \in \mathcal{S}$ are called stabilizers.

$\mathcal{S}$ defines a subspace $\mathcal{C} \subset (\mathbb{C}^2)^{\otimes n}$,

$$\mathcal{C} := \{ |\psi\rangle \mid |\psi\rangle = S|\psi\rangle \ \forall S \in \mathcal{S} \},$$

the code space of a stabilizer code.

$\mathcal{S}$ can also be characterized by a minimal set of generators $S_1, \dots, S_r \in \mathcal{S}$.

Lemma: $\dim \mathcal{C} = 2^{n-r}$.

Proof: (sketch!)

i) S_1 has same # of ± 1 eigenvalues (tr $S_1 = 0$)

$\Rightarrow$ split space in half.

$\Pi_1 = \frac{1}{2}(I + S_1)$: proj. on $+1$ eigenspace of S_1.

ii) $\Pi_1 S_2 = S_2 \Pi_1$ (as $S_1 S_2 = S_2 S_1$),

and $\Pi_1 S_2 \Pi_1 = \frac{1}{2}(I + S_1) S_2$

± 1 - eigensp. of S_2 on $+1$ - eigenspace of S_1
(0 on -1 - eigensp. of S_1)

$$\operatorname{tr}\left(\tfrac{1}{2}(I+S_1)S_2\right) = \tfrac{1}{2}\Big(\underbrace{\operatorname{tr}(S_1)}_{=0} + \underbrace{\operatorname{tr}(S_1 S_2)}_{=0\text{: ind. set of gen.-s}}\Big)$$

$$= 0$$

$\Rightarrow$ S_2 has eq. # of $+1/-1$ eigenvals
on $+1$- eigenspace of S_1
$\Rightarrow$ split again in half.

iii) continue inductively! ☒

b) Error correction conditions for stabilizer codes

What about error corr. conditions?

E_α Pauli errors.

$E_\alpha^\dagger E_\beta$ have three possibilities:

i) $E_\alpha^\dagger E_\beta$ anti-comm. with some $S \in \mathcal{S}$:

$$\langle \hat{i} | E_\alpha^\dagger E_\beta | \hat{j} \rangle = \langle \hat{i} | E_\alpha^\dagger E_\beta \underbrace{S | \hat{j} \rangle}_{S|\hat{j}\rangle = |\hat{j}\rangle}$$

$$= -\langle \hat{i} | S E_\alpha^+ E_\beta | \hat{j} \rangle = -\langle \hat{i} | E_\alpha^+ E_\beta | \hat{j} \rangle$$

$$\Rightarrow \langle \hat{i} | E_\alpha^+ E_\beta | \hat{j} \rangle = 0$$

$\Rightarrow$ QECC satisfied $\Rightarrow$ error correctable!

ii) $E_\alpha^+ E_\beta \in \mathcal{S}$:

$$\langle \hat{i} | \underbrace{E_\alpha^+ E_\beta}_{\in \mathcal{S}} | \hat{j} \rangle = \langle \hat{i} | \hat{j} \rangle = \delta_{ij}$$

$\Rightarrow$ QECC satisfied $\Rightarrow$ error correctable!

iii) $E_\alpha^+ E_\beta$ comm. with all $S \in \mathcal{S}$,

but $E_\alpha^+ E_\beta \notin \mathcal{S}$:

$\Rightarrow$ $E_\alpha^+ E_\beta$ acts non-trivially on code space:

it is a logical operator

In particular:

$$\langle \hat{i} | \underbrace{E_\alpha^+ E_\beta | \hat{j} \rangle}_{= |\hat{i}\rangle} = \langle \hat{i} | \hat{i} \rangle \neq c_{\alpha\beta} \delta_{ij}$$

for some $|\hat{i}\rangle, |\hat{j}\rangle$!

$\Rightarrow$ <u>not correctable</u>! ↯

(<u>Diff. intuition</u>: Impossible to tell if given state is $E_\alpha|\hat{i}\rangle$ or $E_\beta|\hat{j}\rangle \Rightarrow$ not correctable!)

<u>Key question</u>: Given a stabilizer code, what is the shortest $E_\alpha^\dagger E_\beta$ (= Pauli product) of that type (here, "short" refers to # of non-trivial Paulis)

c) Example: 3-qubit code

$$\mathcal{C} = \text{span}\{|000\rangle, |111\rangle\}$$

$$\left.\begin{array}{l} S_1 = ZZI \\ S_2 = ZIZ \end{array}\right\} \Rightarrow \mathcal{S} = \{III, ZZI, ZIZ, \underset{S_1 S_2}{\overset{\|}{IZZ}}\}$$

$$k = \underset{n}{\underset{\|}{3}} - \underset{r}{\underset{\|}{2}} = 1 \Rightarrow 1 \text{ encoded qubit}$$

Single-qubit X errors:

$$E_\alpha = III,\ IIX,\ IXI,\ XII$$

$$E_\alpha^\dagger E_\beta = III,\ IIX,\ IXI,\ XII,$$
$$XXI,\ XIX,\ IXX$$

$\Rightarrow$ anti-commu. w/ S_1, S_2, both S_1 & S_2,

or are element of S (for III).

$\Rightarrow$ correctable!

Single-qubit Z errors:

$E_\alpha^\dagger E_\beta = ZII$ is one possibility

But: ZII commu. w/ S_1, S_2, but $ZII \notin S$!

$\Rightarrow$ Z errors not correctable!

Logical operators:

(at the same time: uncorrectable $E_\alpha^\dagger E_\beta$!)

- $\hat{Z} = \underbrace{ZII}_{\text{distance 1 } ↯}$

  – or any $\hat{Z}' = \hat{Z} \cdot S$, $S \in \mathcal{S}$, e.g. IZI, ZZZ, ...

- $\hat{X} = XXX$

  – or e.g. $\hat{X}' = XXX \cdot ZZI = -YYX$, etc ...

Note: $\hat{X}\hat{Z} = -\hat{Z}\hat{X}$ – and this is all we have to require from the logical Pauli operators!

Error detection and correction:

X error E_α can be detected by anti-comm. pattern.

e.g.: • XII anti-comm. w/ ZIZ, $ZZI \in \mathcal{S}$.

• can be measured: $ZZI|\psi\rangle \overset{?}{=} \pm|\psi\rangle$ etc.

$\Rightarrow$ allows to detect error (up to a T s.th.

$TS = ST$ $\forall S \in \mathcal{S}$, and thus $T \in \mathcal{S}$ for correctable errors).

d) More examples:

3-qubit phase flip code:

$$S_1 = XXI$$

$$S_2 = IXX$$

$$\hat{X} = XII$$

$$\hat{Z} = ZZZ$$

9-qubit Shor code:

$$\left.\begin{aligned}
S_1 &= ZZI\ III\ III \\
S_2 &= IZZ\ III\ III \\
S_3 &= III\ ZZI\ III \\
S_4 &= III\ IZZ\ III \\
S_5 &= III\ III\ ZZI \\
S_6 &= III\ III\ IZZ \\
S_7 &= XXX\ XXX\ III \\
S_8 &= III\ \underline{XXX}\ \underline{XXX}
\end{aligned}\right\}$$

8 indep. stabilizers
||
1 encoded qubit

logical X of 3-qubit code!

Logical operators:

e.g.:

$\hat{Z} = ZZZ\,ZZZ\,ZZZ$

$\hat{X} = XXX\,XXX\,XXX$

— these commute w/ S_i, as those have even # of X/Z, but are $\notin \mathcal{S}$, since they have odd # of X/Z.

simpler ("shorter") logical ops:

e.g. $\hat{Z} = ZII\,ZII\,ZII$

$\hat{X} = XXX\,III\,III$

($\Rightarrow$ distance 3!)

Also means that $\hat{X}$ and $\hat{Z}$ can be measured by measuring only 3 qubits!
(But: meas. a general function of $\hat{X}$ & $\hat{Z}$ requires at least 5 qubits because of no-cloning argument!)

<u>Note</u>: 9-qubit code is <u>degenerate</u>:

$S_1 = Z\,I\,I\,I\,I\,I\,I\,I\,I$ and

$S_2 = I\,Z\,I\,I\,I\,I\,I\,I\,I$

have same syndrome, since

$S_1 S_2 = Z\,Z\,I\,I\,I\,I\,I\,I\,I \in \mathcal{S}$.

e) The 5-qubit code

Consider the stabilizer code on 5 qubits w/ generators

$$\left.\begin{aligned} S_1 &= X\,Z\,Z\,X\,I \\ S_2 &= I\,X\,Z\,Z\,X \\ S_3 &= X\,I\,X\,Z\,Z \\ S_4 &= Z\,X\,I\,X\,Z \end{aligned}\right\}$$

encodes $5-4=1$ qubit

<u>cyclic</u> code: $S_1, \dots, S_5$ are cyclic permutations.

$\Rightarrow$ cyclic codewords!

$(S_5 = Z\,Z\,X\,I\,X = S_1 S_2 S_3 S_4)$

<u>Corrects any 1-qubit error</u>:

$E_\alpha^\dagger E_\beta$ = product of ≤ 2 Paulis

$\Longrightarrow$ anti-comm. w/ <u>at least</u> one S_i, $i=1,\dots,5$

(Why? Fix pos. of 1st Pauli, pick S_k which has $\underline{I}$ there. Then, 2nd Pauli must agree with that on S_k; and conversely. But: can check that those choices won't commute w/ some other S_i.)

$\Longrightarrow$ correctable $\Longrightarrow$ $d \geq 3$.

(And $d \leq 3$ from no-cloning: $[[5,1,3]]$-QECC!)

<u>Error syndromes</u> ($1 \equiv$ anti-comm. = eigenval. -1)

	X error on qubit					Z error on					Y error on				
	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
S_1	0	1	1	0	0	1	0	0	1	0	1	1	1	1	0
S_2	0	0	1	1	0	0	1	0	0	1	0	1	1	1	1
S_3	0	0	0	1	1	1	0	1	0	0	1	0	1	1	1
S_4	1	0	0	0	1	0	1	0	1	0	1	1	0	1	1
S_5	1	1	0	0	0	0	0	1	0	1	1	1	1	0	1

15 errors, 15 syndromes $\Longrightarrow$ non-degenerate.

All possible $2^4 - 1 = 15$ syndromes appear.

Logical operators:

$$\hat{Z} = ZZZZZ$$
$$\hat{X} = XXXXX$$

comm. w/ all S_i (even # of X & Z in S_i), but for same reason $\notin S$!

simpler choices:

e.g. $\hat{Z}' = \hat{Z} \cdot S_3 = -YZYII$

$\hat{X}' = \hat{X} \cdot S_2 = -XIYYI$

$\Rightarrow$ distance $d = 3$

& logical info in $\hat{Z}$ or $\hat{X}$ basis can be obtained by meas. only 3 qubits!

(Note: General feature of distance-d code!)

Syndrome meas. + correction can be done using only CNOT, H, X (for corr.), and ancillas.

(Again: gen. feature of stabilizer code: need to compute parity of X & Z eigenvalues.)

f) Epilogue: Clifford circuits & universal q. computation

Definition (Clifford group)

The Clifford group $\mathcal{Cl}_n$ on n qubits consists of all gates which map Paulis to Paulis:

$$\mathcal{Cl}_n = \{ C \text{ unitary} \mid C(P_1 \otimes \dots \otimes P_n)C^\dagger = P_1' \otimes \dots \otimes P_n' \}$$

Theorem (w/out proof):

$$\mathcal{Cl}_n = \{ \text{all circuits built from CNOT, } S = \begin{pmatrix} 1 & \\ & i \end{pmatrix}, H \}$$

These are also called Clifford gates and Clifford circuits.

(i.e.: Any circuit C which maps Paulis to Paulis is of this form!)

Observation: Only $T = \begin{pmatrix} 1 & \\ & e^{i\pi/4} \end{pmatrix}$ missing for a universal gate set!

Remark: Circuits consisting only of Clifford gates, w/ initial state $|0\rangle^{\otimes n}$, and Pauli measurements, can be simulated efficiently (in essence, by describing state in terms of stabilizers).

$\Longrightarrow$ The "simple" gate T gives Clifford circuit universal power!

Key question for error-corrected q. computing:

How can we apply gates on encoded qubits?

Idea I: Decode $\to$ Apply $\to$ Encode:

Not good - protection lost during operation!

Idea II: Can we apply gates directly to encoded quantum information?

$\to$ Focus on stabilizer states!

1. <u>Clifford gates</u> can be applied to encoded qubits:

Clifford gate $\hat{C}$ on <u>logical qubit</u>

$\Updownarrow$

$\hat{C}$ maps logical Paulis to logical Paulis

$\Downarrow$ logical Paulis = products of physical Paulis

$\hat{C}$ maps physical Paulis to physical Paulis

$\Downarrow$

$\hat{C}$ is a Clifford gate on <u>physical qubits.</u>

$\Rightarrow$ Can implement logical $\hat{C}$ directly on physical qubits.

<u>E.g.:</u> $\hat{H}$ gate on 5-qubit code:

$$\left.\begin{aligned} \hat{X} &= XXXXX \\ \hat{Z} &= ZZZZZ \\ \hat{H}\hat{X}\hat{H} &= \hat{Z} \end{aligned}\right\}$$

need to find Clifford gate s.th.

$XXXXX \leftrightarrow ZZZZZ$

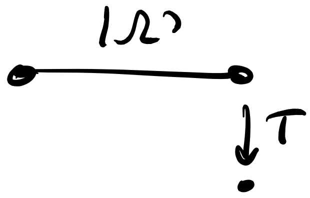
& $\mathcal{S}$ is preserved.

2. Non-Clifford gates:

Can we also realize non-Clifford gates – e.g. $T = \begin{pmatrix} 1 & \\ & e^{i\pi/4} \end{pmatrix}$ – in a robust way?

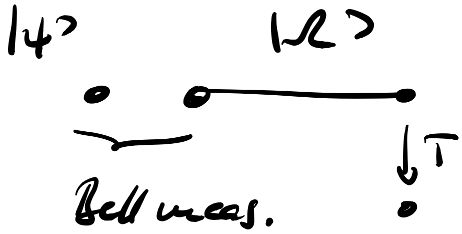
One idea: <u>"Gate teleportation"</u>

i) Prepare $|\chi\rangle = (I \otimes T)|\Omega\rangle$, where $|\Omega\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$|\chi\rangle =$ 

<u>"magic state"</u>

ii) teleport state $|\psi\rangle$ of q. comp. through $|\chi\rangle$:



↳ state on the right is $T \cdot P|\psi\rangle$, with P a Pauli matrix.

↓

Can be transformed to $T|\psi\rangle$ by Clifford gates!

- Teleportation consists only of Clifford gates and meas. in Z basis $\Rightarrow$ can be done on encoded state.

- Encoded $|\chi\rangle$ can be prepared before "offline" (e.g. until success).

$\Rightarrow$ Can carry out <u>universal q. computation</u> on <u>encoded qubits.</u>