Lecture & Proseminar 250120/250122

"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2020/21

— Exercise Sheet #2 —

Problem 1: No-cloning theorem.

a) Show that it is possible to build a cloning device which can copy all computational basis states $\{|i\rangle\}$, i.e., a unitary U such that

$$U|i\rangle|0\rangle = |i\rangle|i\rangle \ .$$

Give an explicit construction of such a U for qubits.

b) Show that such a cloner U can also be built for any other ONB $\{|\phi_i\rangle\}$,

$$U|\phi_i\rangle|0\rangle = |\phi_i\rangle|\phi_i\rangle \ .$$

c) The no-cloning theorem states that there exists no unitary U which implements $U|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle$ for all $|\psi\rangle$. Show that this even holds when we allow for an additional auxiliary system, i.e. there exists no U which implements

$$U|\psi\rangle|0\rangle|0\rangle = |\psi\rangle|\psi\rangle|\gamma_\psi\rangle$$

for any final state $|\gamma_\psi\rangle$ of the auxiliary system (which can depend on $|\psi\rangle$ in any possible way).

Problem 2: Ensemble decompositions by measurement.

a) Consider a state $|\psi\rangle = \alpha|0\rangle_A|0\rangle_B + \beta|1\rangle_A|1\rangle_B$, shared between two parties A and B, with Hilbert space dimensions $d_A = 2$ and $d_B = 4$, respectively. Determine the probabilities p_i and Alice's post-measurement states $|\phi_i\rangle$ if Bob measures in the basis (check that it is an ONB!)

$$(|0\rangle + |2\rangle)/\sqrt{2}\,, \quad (|1\rangle + |3\rangle)/\sqrt{2}\,, \quad (|0\rangle \pm |1\rangle - |2\rangle \mp |3\rangle)/2$$

(note the $\pm$).

What ensemble interpretation of Alice's state does this give? Check that this gives the correct reduced density matrix.

b) Consider the case where Bob's system has a general dimension d_B, and where he measures in a basis

$$|b_i\rangle = \sum u_{ij}|j\rangle \ .$$

i) What properties does the matrix $U = (u_{ij})$ satisfy?

ii) What is the form of the resulting post-measurement ensemble $\{(p_i, |\phi_i\rangle)\}$ for Alice's state?

Problem 3: Bloch sphere for mixed states.

a) Show that any hermitian matrix ρ with $\operatorname{tr}\rho = 1$ can be written as

$$\rho = \frac{I + \vec{r}\cdot\vec{\sigma}}{2}$$

with a vector $\vec{r} \in \mathbb{R}^3$.

b) What are the eigenvalues of ρ? How do they depend on $|\vec{r}|$?

c) What property do the points $\vec{r}$ for which $\rho \geq 0$ satisfy?

d) Interpret this in terms of the Bloch sphere: What do points on the inside or outside of the Bloch sphere correspond to? What about the surface of the sphere?

e) What is the interpretation of the point at the center of the Bloch sphere? What is the interpretation of points along the z axis? How does this generalize to other points inside the Bloch sphere?

f) What is the location of a state $\rho = p|\psi\rangle\langle\psi| + (1-p)|\phi\rangle\langle\phi|$ in the Bloch sphere? How does this generalize to general convex combinations $\rho = \sum p_i|\psi_i\rangle\langle\psi_i|$?