Lecture & Proseminar 250120/250122
"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2020/21

— Exercise Sheet #3 —

Problem 1: Evolution and measurement of density matrices in the ensemble interpretation.

Use the ensemble interpretation

$$\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$$

of a density matrix ρ to derive

1. how it evolves under the action of a unitary U,
2. the outcome probabilities for a measurement with projections $\{E_n\}$, and
3. the post-measurement states after such a measurement.

(That is, consider ρ as a probabilistic mixture of pure states $|\psi_i\rangle$ with probabilities p_i, and use what you know about evolution and measurement of pure states.)
Show that the result obtained is independent of the chosen ensemble decomposition.

Problem 2: Ambiguity of ensemble decomposition.

Complete the proof given in the lecture for the relation

$$\sqrt{p_i}|\psi_i\rangle = \sum u_{ij}\sqrt{q_j}|\phi_j\rangle$$

of different ensemble decompositions

$$\rho = \sum p_i|\psi_i\rangle\langle\psi_i| = \sum q_j|\phi_j\rangle\langle\phi_j| \ .$$

1. Show that any ensemble decomposition must have at least as many terms as the eigenvalue decomposition $\rho = \sum \lambda_k|e_k\rangle\langle e_k|$.
2. Show that the proof from the lecture extends to the case where the other decomposition has more terms than the eigenvalue decomposition, to show

   $$\sqrt{p_i}|\psi_i\rangle = \sum u_{ik}\sqrt{\lambda_k}|e_k\rangle \ . \qquad (*)$$

   What property does this imply for $U = (u_{ik})$?
3. Show that that the relation $(*)$ can be inverted to give a formula for $\sqrt{\lambda_k}|e_k\rangle$.
4. Now consider the case where neither of the two ensembles is an eigenvalue decomposition. Use the fact that there are $U = (u_{ik})$ and $V = (v_{jk})$ which connect them to the eigenvalue decomposition to derive the general relation between two ensemble decompositions of a given state ρ. What is the form of the transformation matrix $W = (w_{ij})$ in terms of U and V? What properties do $W^\dagger W$ and $WW^\dagger$ satisfy?

Problem 3: SVD from eigenvalue decomposition

In this problem, we will construct the singular value decomposition from the eigenvalue decomposition. To this end, consider a general rectangular matrix M of size $m \times n$, $m \leq n$.

1. Consider the eigenvalue decomposition of $MM^\dagger$, $MM^\dagger = U\Lambda U^\dagger$, with U unitary and Λ diagonal. What property do the eigenvalues, i.e. the entries of Λ, satisfy?

2. Define $D = \sqrt{\Lambda}$ (since Λ is diagonal, this is the square root of the diagonal elements). Consider first the case where all diagonal elements of D are zero. Let

$$V := M^\dagger U D^{-1} \ .$$

   (a) What is $V^\dagger V$?

   (b) What is $UDV^\dagger$?

3. Generalize this argument to the case where $M^\dagger M$ has zero eigenvalues.