**Lecture & Proseminar 250120/250122**
**"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2020/21**

**— Exercise Sheet #4 —**

**Problem 1: Measurements and filtering**
Suppose that a bipartite system $AB$ is initially in the state

$$|\phi_\lambda\rangle = \sqrt{\lambda}|00\rangle + \sqrt{1-\lambda}|11\rangle \ .$$

The goal of Alice and Bob is to obtain a maximally entangled state

$$|\Omega\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

with some probability by applying local operations only. Specifically, the plan is that Alice will apply a POVM measurement to achieve that.

1. Show that the operators $M_0 = (|0\rangle\langle 0| + \sqrt{\gamma}|1\rangle\langle 1|)_A \otimes I_B$ and $M_1 = \sqrt{1-\gamma}|1\rangle\langle 1|_A \otimes I_B$, with $0 \le \gamma \le 1$, define a POVM measurement. (Note that these describe measurements carried out on Alice's side only!)
2. Determine the outcome probabilities and the post-measurement states for both measurement outcomes.
3. Find a value $\gamma$ such that one of post-measurement states becomes a maximally entangled state. Calculate the corresponding probability with which the initial state becomes a maximally entangled state.
4. In the lecture, we have shown that any POVM measurement can be implemented by adding an auxiliary system in state $|0\rangle$, applying a unitary, and measuring the auxiliary system in the computational basis. Construct such a unitary for the POVM of Alice above.

**Problem 2: SIC-POVMs**

A *symmetric informationally complete POVM* (SIC-POVM) in $d$ dimensions is a POVM $\{F_i\}_{i=1,\dots,d^2}$ consisting of $d^2$ operators $F_i = \lambda \Pi_i$, where the $\Pi_i = |\phi_i\rangle\langle\phi_i|$ are rank-1 projectors $\Pi_i^2 = \Pi_i$, such that

i) $\sum_{i=1}^{d^2} F_i = I$ (i.e. the $F_i$ form a POVM), and

ii) $\mathrm{tr}(F_i F_j) = K$ for $i \neq j$, where $K$ is independent of $i$ and $j$ (that is, the POVM is *symmetric*).

1. Use the two conditions (i) and (ii) to determine the values of $\lambda$ and $K$.
2. Now consider $d = 2$ (qubits). Consider four states $|\phi_i\rangle$ sitting at the four corners of a tetrahedron. (Any tetrahedron is good, but it might be convenient to have one corner along the $z$ axis and another one in the $x$-$z$-plane.) Derive the form of $|\phi_i\rangle$, and show that they give rise to a SIC-POVM (following the convention above).
3. Show that the operators $\{F_i\}$ of a SIC-POVM (with the conditions (i) and (ii) above, for arbitary $d$) are linearly independent. (*Easier version:* Show this only for the qubit SIC-POVM constructed in point 2.)
4. Show that the linear independence of the $\{F_i\}$ implies that there exist $G_i$ such that we can write

   $$\rho = \sum_{i=1}^{d^2} G_i \mathrm{tr}[F_i \rho]$$

   – that is, the POVM is *informationally complete*, i.e., we can reconstruct any state $\rho$ from the outcome probabilities of the POVM. What is the form of the $G_i$?