Lecture & Proseminar 250120/250122
"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2020/21

— Exercise Sheet #5 —

Problem 1: CHSH inequality I: No-signalling correlations.
Consider the scenario of the CHSH inequality. Let

$$\langle C\rangle = \langle a_0 b_0\rangle + \langle a_1 b_0\rangle + \langle a_0 b_1\rangle - \langle a_1 b_1\rangle \ . \tag{1}$$

Here, $a_x = \pm 1$ and $b_y = \pm 1$ are the outcomes obtained by Alice and Bob given an input (measurement setting) of x (on Alice's side) and y (on Bob's side). As explained in the lecture, the physical theory underlying the CHSH scenario is given by some joint conditional probability distribution $P(a, b|x, y)$ [that is, it has to satisfy $0 \le P(a, b|x, y) \le 1$ and $\sum_{a,b} P(a, b|x, y) = 1$]. The expectation value $\langle a_x b_y\rangle$ is then given by

$$\langle a_x b_y\rangle = \sum_{a,b,x,y} a\, b\, P(a, b|x, y) \ .$$

1. A non-signalling is a distribution which does not allow for communication between Alice and Bob, i.e., Alice's marginal distribution $P^A(a|x) = \sum_b P(a, b|x, y)$ does not depend on Bob's input y, and vice versa. Show that no-signalling distributions can obtain the maximum possible value $|\langle C\rangle| = 4$.
2. Give a distribution $P(a, b|x, y)$ which violates no-signalling.

Problem 2: CHSH inequality II: Tsirelson's bound.
Tsirelson's inequality bounds the largest possible violation of the CHSH inequality (1) in quantum mechanics (namely $2\sqrt{2}$). To this end, let a_0, a_1, b_0, b_1 be Hermitian operators with eigenvalues ± 1, so that

$$a_0^2 = a_1^2 = b_0^2 = b_1^2 = I \ .$$

Here, a_0 and a_1 describe the two measurements of Alice, and b_0 and b_1 those of Bob; in particular, this means that Alice's and Bob's measurements commute, i.e. $[a_x, b_y] = 0$ for all $x, y = 0, 1$. Define

$$C = a_0 b_0 + a_1 b_0 + a_0 b_1 - a_1 b_1 \ .$$

1. Determine C^2.
2. The (operator) norm of a bounded operator M is defined by

$$\|M\| = \sup_{|\psi\rangle} \frac{\|M|\psi\rangle\|}{\||\psi\rangle\|} \ ,$$

   that is, the norm of M is the maximum eigenvalue of $\sqrt{M^\dagger M}$. Verify that the norm has the properties

$$\|MN\| \le \|M\|\|N\| \ ,$$
$$\|M + N\| \le \|M\| + \|N\| \ .$$

   Also, what is the operator norm of a hermitian operator in terms of its eigenvalues?
3. Find an upper bound on the norm $\|C^2\|$.
4. Show that for Hermitian operators $\|C^2\| = \|C\|^2$. Use this to obtain an upper bound on $\|C\|$.
5. Explain how this inequality gives a bound on the maximum possible violation of the CHSH inequality in quantum mechanics. This is known as Tsirelson's bound, or Tsirelson's inequality.

Problem 3: Quantum channels.
In this problem, we will study some commonly appearing quantum channels. In addition to the problems listed, verify for each channel that it is a CPTP map (completely positive trace preserving map) and give its Kraus representation.

1. *Dephasing channel.* This channel acts as

$$\mathcal{E}(\rho) = (1-p)\,\rho + p\,Z\rho Z\ .$$

   Show that the action of the dephasing channel on the Bloch vector is

$$(r_x, r_y, r_z) \mapsto ((1-2p)r_x, (1-2p)r_y, r_z)\ ,$$

   i.e., it preserves the component of the Bloch vector in the Z direction, while shrinking the X and Y component.

2. *Amplitude damping channel.* The amplitude damping channel is giving by the Kraus operators

$$M_0 = \sqrt{\gamma}|0\rangle\langle 1|, \quad M_1 = |0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1|\ ,$$

   where $0 \leq \gamma \leq 1$. Here, M_0 describes a decay from $|1\rangle$ to $|0\rangle$, and γ corresponds to the decay rate.

   (a) Consider a single-qubit density operator with the following matrix representation with respect to the computation basis

$$\rho = \begin{pmatrix} 1-p & \eta \\ \eta^* & p \end{pmatrix},$$

   where $0 \leq p \leq 1$ and η is some complex number. Find the matrix representation of this density operator after the action of the amplitude damping channel.

   (b) Show that the amplitude damping channel obeys a composition rule. Consider an amplitude damping channel $\mathcal{E}_1$ with parameter γ_1 and consider another amplitude damping channel $\mathcal{E}_2$ with parameter γ_2. Show that the composition of the channels, $\mathcal{E} = \mathcal{E}_1 \circ \mathcal{E}_2$, $\mathcal{E}(\rho) = \mathcal{E}_1(\mathcal{E}_2(\rho))$, is an amplitude damping channel with parameter $1-(1-\gamma_1)(1-\gamma_2)$. Interpret this result in light of the interpretation of the γ's as a decay probability.

3. *Twirling operation.* Twirling is the process of applying a random Pauli operator (including the identity) with equal probability. Explain why this corresponds to the channel

$$\mathcal{E}(\rho) = \tfrac{1}{4}\rho + \tfrac{1}{4}X\rho X + \tfrac{1}{4}Y\rho Y + \tfrac{1}{4}Z\rho Z\ .$$

   Show that the output of this channel is the maximally mixed state for any input, $\mathcal{E}(\rho) = \frac{1}{2}I$.

   Hint: Represent the density operator as $\rho = \frac{1}{2}(I + r_x X + r_y Y + r_z Z)$ and apply the commutation rules of the Pauli operators.