Lecture & Proseminar 250120/250122
"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2020/21

— Exercise Sheet #7 —

Problem 1: Majorization
In this problem, we prove that $x \prec y$ implies that $x = \sum_j q_j P_j y$ for some probability distribution q_j and permutation matrices P_j, where $x, y \in \mathbb{R}^d_{\geq 0}$. The proof will proceed by induction in the dimension d of the space.

1. Let $x, y \in \mathbb{R}^d_{\geq 0}$, $x \prec y$, and let the entries of x and y (denoted by x_k, y_k) be ordered descendingly.
2. Show that there exist k and $t \in [0, 1]$ such that $x_1 = ty_1 + (1 - t)y_k$. For which k does this work? For the following steps, we choose the *smallest such* k.
3. Define $D = tI + (1 - t)T$, where T is the permutation matrix which transposes the 1st and k-th matrix elements. What are the components of the vector Dy?
4. Define x' and y' by eliminating the first entry from x and Dy, respectively. Show that $x' \prec y'$.
5. Show that this way, we can inductively prove the claim.

Problem 2: Fidelity.

1. Prove that for normalized vectors $|\psi\rangle$ and $|\phi\rangle$,
$$\big|\langle\psi|O|\psi\rangle - \langle\phi|O|\phi\rangle\big| \leq \sqrt{8}\,\sqrt{1 - |\langle\psi|\phi\rangle|}\,\|O\|_\infty\ ,$$
with $\|O\|_\infty = \|O\|_{\mathrm{op}} = \sup_{|\psi\rangle} \frac{\|O|\psi\rangle\|}{\||\psi\rangle\|}$. Use this to prove
$$\big|\langle\psi|O|\psi\rangle - \langle\phi|O|\phi\rangle\big| \leq 2\sqrt{\delta}\|O\|_\infty\ , \qquad (*)$$
to leading order in δ, where $\delta = 1 - F$, with $F = |\langle\psi|\phi\rangle|^2$ the fidelity.
2. Use the operator Hölder inequality
$$|\mathrm{tr}(AB)| \leq \|A\|_1\|B\|_\infty\ ,$$
where the *trace norm* $\|A\|_1$ is the sum of the singular values of A (i.e. for hermitian A the sum of the absolute value of the eigenvalues) to prove $(*)$ directly (and without the need for a leading-order approximation).

Problem 3: Multi-copy protocols.
Consider $|\chi\rangle = \sqrt{\frac{2}{3}}|00\rangle + \sqrt{\frac{1}{3}}|11\rangle$, and $|\Phi^+\rangle = \sqrt{\frac{1}{2}}(|00\rangle + |11\rangle)$.

1. Show that the optimal average yield per copy, $\bar{p} = (p_1 + 2p_2)/2$, for the conversion of $|\chi\rangle^{\otimes 2}$ to $|\Phi^+\rangle^{\otimes 2}$ and $|\Phi^+\rangle$ with probabilities p_2 and p_1, respectively, does not improve over the single-copy protocol.
2. Show that the average yield for the conversion of $|\chi\rangle^{\otimes 3}$ into one, two, or three copies, $\bar{p} = (p_1 + 2p_2 + 3p_3)/3$, improves over the single-copy protocol.
3. Now let $|\Phi_d\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^d |i,i\rangle$, and consider a protocol where we convert $|\chi\rangle^{\otimes 2}$ into $|\Phi_2\rangle$, $|\Phi_3\rangle$, and $|\Phi_4\rangle$ with probabilities p_2', p_3', and p_4'. Show that if we assign the entanglement $\log d$ (with the log base 2) to $|\Phi_d\rangle$, we *can* improve the average yield $\bar{p} = (p_2' + (\log 3)\, p_3' + 2p_4')/2$ over the single-copy protocol.