Lecture & Proseminar 250120/250122
"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2020/21

— Exercise Sheet #8 —

Problem 1: Decay of entanglement.

Consider a Bell state $\rho = |\Phi^+\rangle\langle\Phi^+|$, where $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Superposition states like ρ are typically not stable, but decay over time. A typical evolution is that the populations (i.e., the diagonal elements) becom equal, while the off-diagonal elements decay exponentially to zero. Suppose that the state evolves as

$$\rho(t) = p_+|00\rangle\langle 00| + p_-|01\rangle\langle 01| + p_-|10\rangle\langle 10| + p_+|11\rangle\langle 11| + \tfrac{1}{2}e^{-t/T_2}|00\rangle\langle 11| + \tfrac{1}{2}e^{-t/T_2}|11\rangle\langle 00| \ ,$$

with $p_\pm = \frac{1}{4}(1 \pm e^{-t/T_1})$.

1. Give the matrix form of $\rho(t)$.
2. What is the limit $\lim_{t\to\infty} \rho(t)$? Is it entangled?
3. Take the partial transpose $\rho(t)^{T_B}$ and give its matrix form.
4. Calculate the eigenvalues of $\rho(t)^{T_B}$.
5. Sketch how the eigenvalues change over time for $T_1 = T_2 = 1$. What it the asymptotic limit?
6. Determine and plot the negativity $\mathcal{N}(\rho(t))$ and log-negativity $E_N(\rho(t))$ as a function of time.
7. Find the time t_{sep} after which $\rho(t_{\text{sep}})$ becomes separable.

Problem 2: Bell inequalities and witnesses.

The CHSH operator – that is, the operator measured in the CHSH inequality – can be written as

$$C = \vec{n}_1\vec{\sigma} \otimes \vec{n}_0\vec{\sigma} + \vec{n}_1\vec{\sigma} \otimes \vec{n}_2\vec{\sigma} + \vec{n}_3\vec{\sigma} \otimes \vec{n}_2\vec{\sigma} - \vec{n}_3\vec{\sigma} \otimes \vec{n}_0\vec{\sigma}$$

with $\vec{n}_k = (\cos(k\pi/4), 0, \sin(k\pi/4))$. Then, the CHSH inequality states that $|\mathrm{tr}[C\rho]| \leq 2$ for all ρ which are described by a local hidden variable (LHV) model.

1. Show that the measurement of C on any separable state $\rho = \sum p_i \rho_i^A \otimes \rho_i^B$ can be described by an LHV model.
2. Use C to construct an entanglement witness W. Provide an explicit form of the witness.
3. In which range of λ does this witness detect Werner states $\rho(\lambda) = \lambda|\Psi^-\rangle\langle\Psi^-| + \frac{1-\lambda}{4}I$, with $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$? How does it compare to the entanglement witness $W = \mathbb{F}$ discussed in the lecture?

Problem 3: Witnesses and reduction criterion.

Consider $W = \mathbb{I} - d|\Omega\rangle\langle\Omega|$, with $|\Omega\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d} |i,i\rangle$.

1. Show that $\mathrm{tr}[W\rho] \geq 0$ for separable states ρ, i.e., W is an entanglement witness.
2. Consider the family
$$\rho_{\text{iso}}(\lambda) = \lambda\,\frac{\mathbb{I}}{d^2} + (1-\lambda)|\Omega\rangle\langle\Omega|$$
of *isotropic states*. In which range of λ is $\rho_{\text{iso}}(\lambda) \geq 0$? In which range of λ does W detect that $\rho_{\text{iso}}(\lambda)$ is entangled?

3. Consider the case $d = 2$. What does W do on the antisymmetric state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$?
4. Derive the positive map Λ corresponding to the witness W. Prove directly that it is indeed a positive map.
5. In which range of λ does Λ detect that $\rho_{\text{iso}}(\lambda)$ is entangled? What does Λ do on the antisymmetric state?