4. POVM measurements

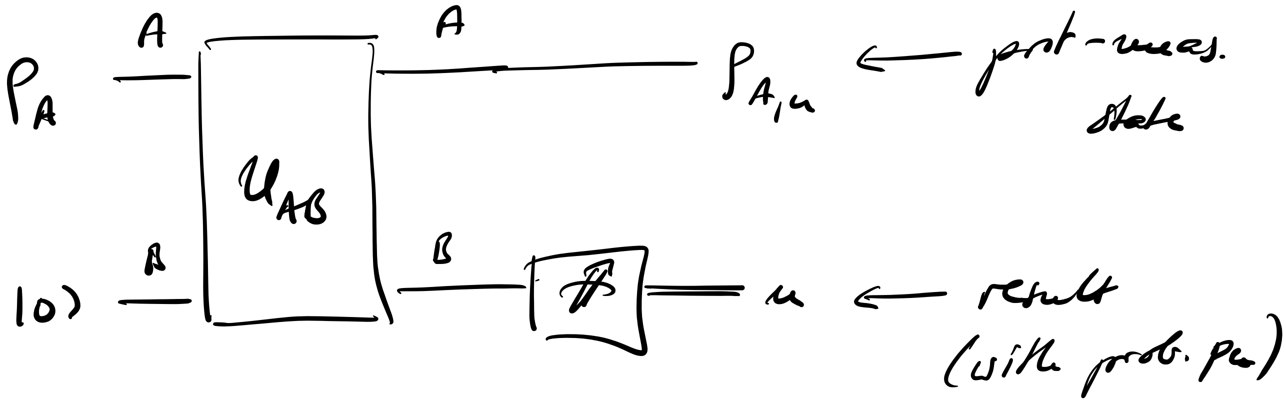
Seen previously: Adding a 2nd system B gives more rich situation.

Thus natural question: What measurements can we do by adding an extra system?

Idea: i) Add auxiliary system ("ancilla") B in state $|0\rangle$

ii) Act w/ unitary U_{AB} on system + ancilla

iii) measure B in $\{|0\rangle, |1\rangle, \dots, |d_B - 1\rangle\}$



Analyze scheme:

Post-meas. state (unnormalized) is:

$$\tilde{\rho}_n^A = \langle n|_B \, U \left(\rho_A \otimes |0\rangle\langle 0|_B\right) U^\dagger |n\rangle_B$$

$$= \langle \mu|_B \, \mathcal{U} |0\rangle_B \;\; \rho_A \;\; \langle 0|_B \, \mathcal{U}^\dagger |\mu\rangle_B$$

$$= \Pi_\mu \, \rho_A \, \Pi_\mu^\dagger ,$$

where we have defined

$$\Pi_\mu := \langle \mu|_B \, \mathcal{U} |0\rangle_B \equiv \left(I_A \otimes \langle \mu|_B \right) \mathcal{U} \left(I_A \otimes |0\rangle_B \right)$$

Then, $p_\mu = tr \, \tilde{\rho}_\mu^A = tr\left(\Pi_\mu \rho_A \Pi_\mu^\dagger\right) = tr\left(\Pi_\mu^\dagger \Pi_\mu \rho_A\right)$,

is the probability for outcome μ,

and $\rho_\mu^A = \frac{1}{p_\mu} \tilde{\rho}_\mu^A$ the post-measurement state.

It holds that

$$\sum \Pi_\mu^\dagger \Pi_\mu = \sum_\mu \langle 0|_B \, \mathcal{U}^\dagger |\mu\rangle_B \langle \mu|_B \, \mathcal{U} |0\rangle_B$$

$$= \langle 0|_B \, \mathcal{U}^\dagger \mathcal{U} |0\rangle_B$$

$$= I_A ,$$

and further $\Pi_\mu^\dagger \Pi_\mu \geq 0$.

(Note: The former implies

$\sum p_\mu = \sum tr\left(\Pi_\mu^\dagger \Pi_\mu \rho\right) = tr\left(\sum \Pi_\mu^\dagger \Pi_\mu \rho\right) = tr(\rho) = 1$).

Definition: A set $\{F_m\}$ of operators, $F_m \geq 0$, $\sum F_m = I$, is called a positive operator-valued measure (POVM).

Note: $F_m := M_m^\dagger M_m$ forms a POVM. If we only care about the post-meas. prob. $p_m = tr(F_m \rho)$, then the measurement is fully characterized by the POVM $\{F_m\}$.

Definition: A POVM measurement is given by a set of operators $\{M_m\}$ with $\sum M_m^\dagger M_m = I$, with outcome probabilities $p_m = tr(M_m^\dagger M_m \rho)$ and post-measurement states $\rho_m = \frac{1}{p_m} M_m \rho M_m^\dagger$.

Alternative Definition: A POVM measurement is given by a set of operators $\{F_m\}$, $F_m \geq 0$, $\sum F_m = I$, with outcome probabilities $p_m = tr(F_m \rho)$.

Relation of the two definitions, & with the initial unitary + ancilla construction:

i) Can any $F_n \geq 0$ be written as $F_n = \Pi_n^\dagger \Pi_n$?

Yes – e.g., take $\Pi_n = \sqrt{F_n}$.

(Unique up to isometric degree of freedom, since

$\Pi_n = U_n \sqrt{\Pi_n^\dagger \Pi_n}$ (the <u>polar decomposition</u>).

ii) Can any POVM meas. $\{\Pi_n\}_{n=0}^{N-1}$, $\sum_{n=0}^{N-1} \Pi_n^\dagger \Pi_n = I$, be realized via ancilla + unitary?

$$X := \begin{pmatrix} \Pi_0 \\ \Pi_1 \\ \vdots \\ \Pi_{N-1} \end{pmatrix}$$

$\sum \Pi_n^\dagger \Pi_n = I \iff X$ has orthogonal columns

$\Longrightarrow$ X can be extended to a unitary U by adding further columns,

$$\mathcal{U} = \begin{array}{c} \\ \langle 0|_B \\ \langle 1|_B \\ \vdots \\ \langle d-1|_B \end{array} \begin{array}{c} \begin{array}{ccc} |0\rangle_B & |1\rangle_B & \cdots \end{array} \\ \begin{pmatrix} \Pi_0 & \cdots & \cdots \\ \Pi_1 & \cdots & \cdots \\ \vdots & & \\ \Pi_{N-1} & \cdots & \cdots \end{pmatrix} \end{array}$$

... this can be understood as a unitary acting on system + ancilla B with dim. $d_B = N$.

$$\Longrightarrow \quad \langle n|_B\, \mathcal{U}\, |0\rangle_B = \Pi_n .$$

$\Longrightarrow$ Any POVM meas. $\{\Pi_n\}$ can be realized by adding ancilla, doing a unitary $\mathcal{U}$ on system + ancilla, and projectively measuring ancilla in $\{|0\rangle, \dots, |d_B-1\rangle\}$ basis.

This is also known as Naimark's Theorem.

Note: The "old-style" measurements where the $\Pi_n = E_n$ (or equivalently $F_n \equiv E_n$) are also called projective measurements.

Is this the most general type of measurement?

i) Minimal requirements for q.m. measurements:

Measurements are linear functionals

$$\rho \longmapsto p_n(\rho) ,$$

which map states to outcome probabilities, such that

$$p_n(\rho) \geq 0 \qquad \forall \rho \geq 0, \ tr(\rho) = 1$$

and

$$\sum p_n(\rho) = 1 \qquad \forall \rho \geq 0, \ tr(\rho) = 1.$$

ii) Linear functionals $\rho \mapsto p_n(\rho)$ are of the form $p_n(\rho) = tr(F_n \rho)$.

(E.g. by using a basis where $\rho = |e_i\rangle\langle e_j|$)

unit vectors.

iii) We can w.l.o.g. choose

$$F_n = F_n^{\dagger}.$$

Otherwise, write

$$F_u = \underbrace{\tfrac{1}{2}(F_u + F_u^+)}_{\text{herm. part}} + \underbrace{\tfrac{1}{2}(F_u - F_u^+)}_{\text{anti-herm. part}},$$

and

$$\mathrm{tr}\left((F_u - F_u^+)\rho\right) = \mathrm{tr}(F_u\rho) - \mathrm{tr}(F_u^+\rho)$$

$$= \mathrm{tr}(F_u\rho) - \underbrace{\overline{\mathrm{tr}(F_u\rho^+)}}_{\rho=\rho^+,\ \text{and}\ \mathrm{tr}(F_u\rho)\geq 0}$$

$$= \mathrm{tr}(F_u\rho) - \mathrm{tr}(F_u\rho) = 0,$$

$$\Rightarrow \mathrm{tr}(F_u\rho) = \mathrm{tr}\left(\tfrac{1}{2}(F_u + F_u^+)\rho\right) \quad \forall \rho \geq 0$$

$\Rightarrow$ Assume from now on that $F_u = F_u^+$.

iv) $1 = \sum p_u(\rho) = \mathrm{tr}\left(\left(\sum F_u\right)\rho\right) \quad \forall \rho \geq 0, \mathrm{tr}\rho = 1$

Let $\{I, H_1, \dots, H_{d^2-1}\}$, $H_i = H_i^+$ be ONB; then, $\mathrm{tr}\, H_i = 0$.

$\Rightarrow H_i = c(\rho - \rho')$, $\rho, \rho' \geq 0$, $\mathrm{tr}\rho = \mathrm{tr}\rho' = 1$.

$\Rightarrow \left(\sum F_u, H_i\right)_{HS} = \frac{1}{d}\mathrm{tr}\left(\left(\sum F_u\right) H_i\right) = 0 \quad \forall i,$

H-S scalar product

$$\left(\sum F_n, I\right)_{HS} = \frac{1}{d}\,\mathrm{tr}\left(\sum F_n\right) = \mathrm{tr}\Big(\big(\sum F_n\big)\underbrace{\tfrac{1}{d}I}_{=\rho}\Big) = 1$$

$$\Rightarrow \quad \sum F_n = I.$$

v) $0 \leq p_n(\rho) = \mathrm{tr}\,(F_n\,\rho) \Rightarrow F_n \geq 0$

(otherwise F_n has a negative eigenvalue

$\lambda < 0$, $\lambda|\phi\rangle = F_n|\phi\rangle$, and then

$\mathrm{tr}\,(F_n|\phi\rangle\langle\phi|) = \lambda < 0$ ↯)

Thus: POVM measurement is the most general linear measurement on density matrices.