

III. Entanglement

1. Introduction

Consider a bipartite state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.

We say that $|\psi\rangle$ is a product state if it can be written as

$$|\psi\rangle = |\phi^A\rangle \otimes |\phi^B\rangle$$

for some $|\phi^A\rangle \in \mathcal{H}_A$, $|\phi^B\rangle \in \mathcal{H}_B$.

If $|\psi\rangle$ is not a product state, we say that

$|\psi\rangle$ is entangled.

How can we tell if a state is a product state or entangled?

Consider Schmidt decomposition of $|\psi\rangle$:

$$|\psi\rangle = \sum_{i=1}^r \lambda_i |\phi_i^A\rangle \otimes |\phi_i^B\rangle$$

↑ ↑
orthonormal

with r the Schmidt rank.

- $r=1 \Rightarrow |\psi\rangle = |\phi_1^A\rangle \otimes |\phi_1^B\rangle \Rightarrow |\psi\rangle$ product.
- $|\psi\rangle$ product $\Rightarrow |\psi\rangle = |\phi^A\rangle \otimes |\phi^B\rangle$ is Schmidt decomposition $\Rightarrow r=1$.

i.e.: Product states have Schmidt rank $r=1$.

Entangled states have Schmidt rank $r \geq 2$.

In particular, states such as

$$|\psi^-\rangle = \frac{1}{\sqrt{2}} (|0\rangle|1\rangle - |1\rangle|0\rangle)$$

$$\text{or } |\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i, i\rangle$$

are entangled.

Product states: Can describe any state of A & B independently, and state

$$\begin{aligned} |\psi\rangle = |\phi^A\rangle \otimes |\phi^B\rangle &\longmapsto (\pi_A \otimes N_B) |\psi\rangle \\ &= \pi_A |\phi^A\rangle \otimes N_B |\phi^B\rangle \end{aligned}$$

stays a product state.

Entangled states:

Generally cannot describe actions of A & B independently - e.g., meas. outcomes in Schmidt basis will be perfectly correlated.

Can we use these correlations for non-trivial tasks?

Questions in entanglement theory:

- How non-classical are entangled states?
- What can we do with entangled states?
(i.e., are they a resource?)
- How can we quantify the amount of entanglement (e.g. in terms of usefulness)?
- Are there different types of entangled states?
- How can we manipulate entangled states?
- What about mixed state entanglement?