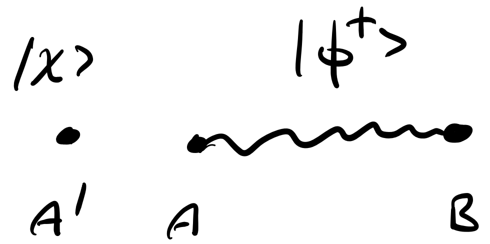
3. Applications of entanglement: Teleportation and dense coding

a) Teleportation

Setup:



- A & B share entangled state $|\phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB}\right)$

- A has unknown quantum state

$$|\chi\rangle_{A'} = a|0\rangle_{A'} + b|1\rangle_{A'}$$

(Could e.g. also be part of a larger system → linearity!)

- A & B cannot (reliably*) transmit quantum states, but can communicate classically "for free".

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* If the line is unreliable, A&B can still use it to create entangled states $|\phi^+\rangle$, e.g. by repeat-until-success, or entanglement distillation (→ later!), or using "quantum repeaters" (→ later!)

Question: Can A get $|\chi\rangle$ (safely) to B?

Problem: Any measurement of $|\chi\rangle$ would only reveal partial information, yet destroy state!

Solution: Quantum Teleportation!

Teleportation Protocol:

(1) A performs measurement on $A'A$ in Bell basis

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$$|\phi^-\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) = (Z \otimes I)|\phi^+\rangle = (I \otimes Z)|\phi^+\rangle$$

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right) = (X \otimes I)|\phi^+\rangle = (I \otimes X)|\phi^+\rangle$$

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right) = (\cancel{t}X \otimes I)|\phi^+\rangle = (I \otimes XZ)|\phi^+\rangle$$

We also write the four Bell states as

$$|\phi_{\alpha\beta}\rangle = (Z^\alpha X^\beta \otimes I)|\phi^+\rangle = (I \otimes X^\beta Z^\alpha)|\phi^+\rangle$$

$\alpha, \beta = 0, 1$

Outcome probabilities for meas. outcome $|\phi_{\alpha\beta}\rangle$:

$$\rho_A = \mathrm{tr}_B\left[|\phi^+\rangle\langle\phi^+|_{AB}\right] = \frac{1}{2} I_A \longleftarrow \text{state of } A.$$

$$p_{\alpha\beta} = \langle\phi_{\alpha\beta}|\ |\chi\rangle\langle\chi|_{A'} \otimes \frac{1}{2} I_A\ |\phi_{\alpha\beta}\rangle$$

$$= \frac{1}{2}\,\mathrm{tr}\left[\left(|\chi\rangle\langle\chi|_{A'} \otimes I_A\right)\ |\phi_{\alpha\beta}\rangle\langle\phi_{\alpha\beta}|\right]$$

$$= \frac{1}{2}\,\mathrm{tr}_{A'}\left[\ |\chi\rangle\langle\chi|_{A'} \cdot \underbrace{\mathrm{tr}_A\left[|\phi_{\alpha\beta}\rangle\langle\phi_{\alpha\beta}|\right]}_{=\frac{1}{2} I_{A'}}\right]$$

$$= \frac{1}{2}\,\mathrm{tr}\left[|\chi\rangle\langle\chi|_{A'} \cdot \frac{1}{2} I_{A'}\right]$$

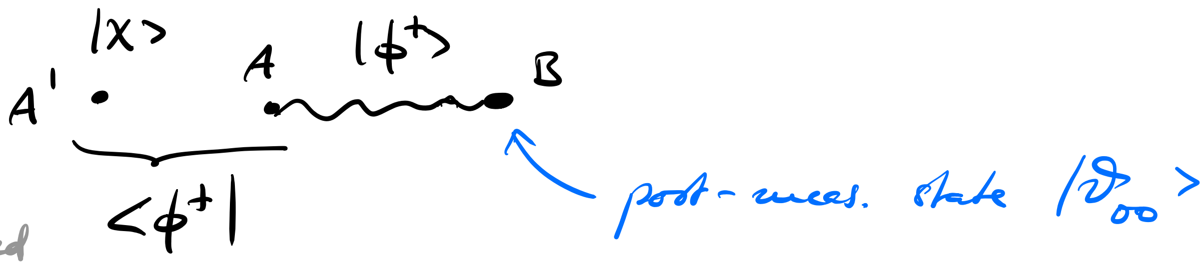
$$= \frac{1}{4}$$

$\Rightarrow$ equal probability $p_{\alpha\beta} = 1/4$ for all outcomes.

(This is good - if $p_{\alpha\beta}$ would depend on $|\chi\rangle$, it would reveal information on $|\chi\rangle$ and thus perturb the state!)

What is the state of B after the measurement?

i) Outcome $|\phi^+\rangle = |\phi_{00}\rangle$:



unnormalized

$$|\tilde{\vartheta}_{00}\rangle = \langle\phi^+|_{A'A} \left(|\chi\rangle_{A'} \otimes |\phi^+\rangle_{AB} \right)$$

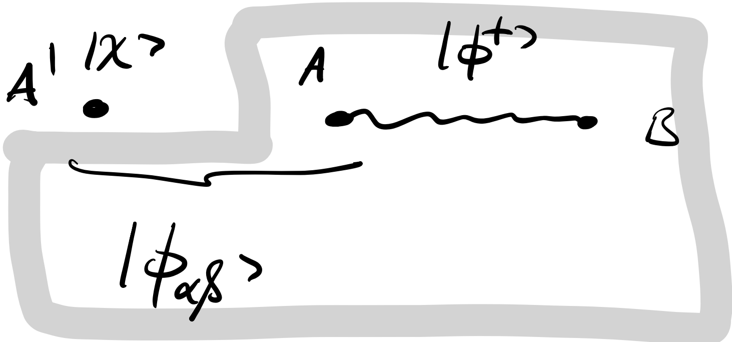
$$= \frac{1}{2} \underbrace{\left(\langle 00|_{A'A} + \langle 11|_{A'A} \right) \left(\left(a|0\rangle_{A'} + b|1\rangle_{A'} \right)\right.}_{= a\langle 0|_A + b\langle 1|_A} \left.\otimes \left(|00\rangle_{AB} + |11\rangle_{AB} \right) \right)$$

$$= \frac{1}{2} \left(a|0\rangle_B + b|1\rangle_B \right) \qquad (*)$$

$\Rightarrow$ State $|\vartheta_{00}\rangle = |\chi\rangle$ appears at B!

(works with 25% probability.)

ii) What about the other outcomes?



First consider $\langle\phi_{\alpha\beta}|_{A'A}|\phi^+\rangle_{AB}$ — worked

going above:

$$\langle\phi_{\alpha\beta}|_{A'A}|\phi^+\rangle_{AB} = \langle\phi^+|_{A'A}\left(I_{A'}\otimes Z_A^\alpha X_A^\beta\right)|\phi^+\rangle_{AB}$$

$$= \langle\phi^+|_{A'A}\left(Z_A^\alpha X_A^\beta \otimes I_B\right)|\phi^+\rangle_{AB}$$

$$= \langle\phi^+|_{A'A}\left(I_A \otimes X_B^\beta Z_B^\alpha\right)|\phi^+\rangle_{AB}$$

$$= X_B^\beta Z_B^\alpha \langle\phi^+|_{A'A}|\phi^+\rangle_{AB}$$

Now combine with derivation ⊛ in part i)

$$|\tilde{\vartheta}_{\alpha\beta}\rangle = \langle\phi_{\alpha\beta}|_{A'A}\left(|\chi\rangle_{A'}\otimes|\phi^+\rangle_{AB}\right)$$

$$= X_B^\beta Z_B^\alpha \underbrace{\langle\phi^+|_{A'A}\left(|\chi\rangle_{A'}\otimes|\phi^+\rangle_{AB}\right)}_{\overset{⊛}{=}\frac{1}{2}|\chi\rangle_B}$$

$$= \underline{\underline{\frac{1}{2}X^\beta Z^\alpha\,|\chi\rangle_B}}\,.$$

$\Rightarrow$ After A's measurement, B obtains $|\vartheta_{\alpha\beta}\rangle = X^\beta Z^\alpha |\chi\rangle$ with probability 1/4 each.

$\Rightarrow$ average state of B — without learning meas. result — is $\frac{1}{4}\sum X^\beta Z^\alpha |\chi\rangle\langle\chi| Z^\alpha X^\beta = \frac{1}{2} I$.

i.e.: Bob has no information about $|\chi\rangle$
(in fact: same state as without meas.)

(2) A communicates meas. outcome (α,β) to B, and
(3) B applies $(X^\beta Z^\alpha)^\dagger$ to their state

$\Rightarrow$ Bob obtains

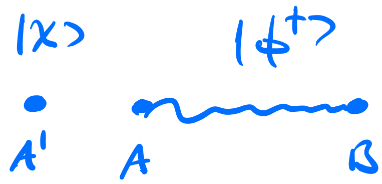
$$(X^\beta Z^\alpha)^\dagger |\vartheta_{\alpha\beta}\rangle = (X^\beta Z^\alpha)^\dagger (X^\beta Z^\alpha)|\chi\rangle = \underline{|\chi\rangle}$$

$\Rightarrow$ Bob obtains $|\chi\rangle$ with probability 1!

$\Rightarrow$ State $|\chi\rangle$ has been teleported to B.

Notes: • No faster-than-light communication
(avg. state of B is $\frac{1}{2}I$ prior to receiving (α,β) — which has finite transm. speed.)

- Communicating 1 qubit requires 1 "ebit" (= a max. entangled state $|\phi^+\rangle$ of 1+1 qubit) + 2 bits of classical communication ("c-bits")



Teleportation protocol – summary:

① Measure A, A' in $|\Phi_{\alpha\beta}\rangle$ basis.

② Communicate (α,β) from A to B.

③ Apply $(X^\beta Z^\alpha)^\dagger$ on B.

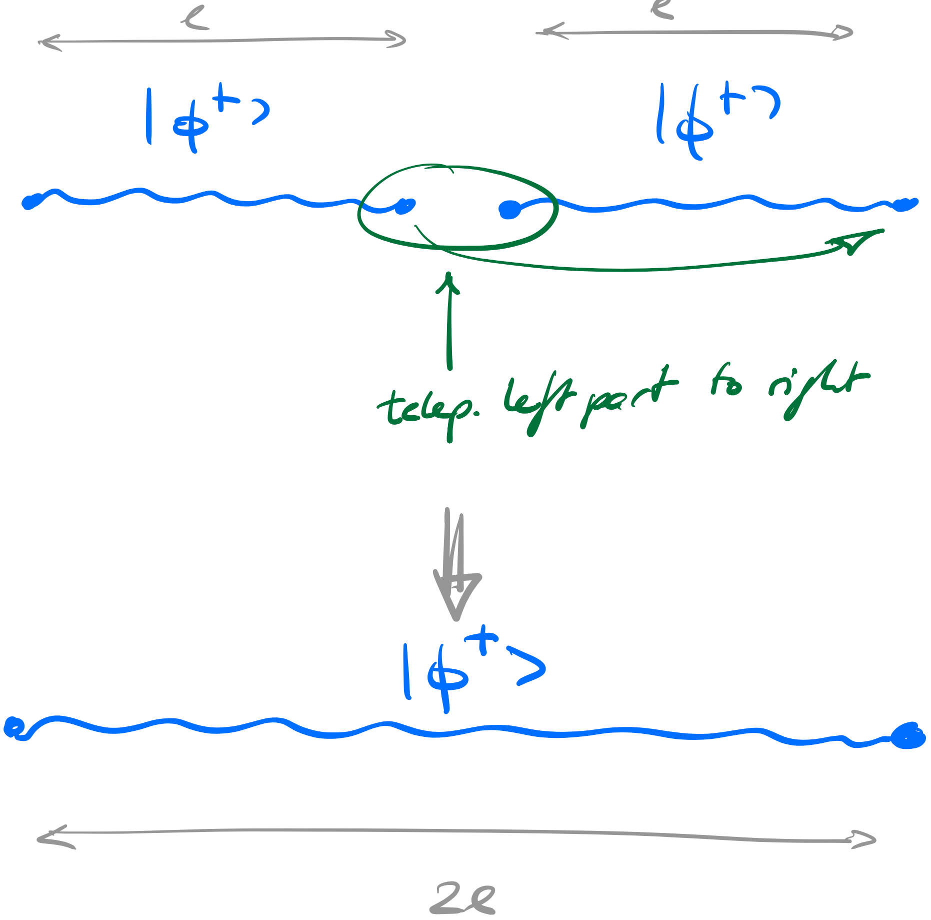
Can be straightforwardly generalized to $\mathbb{C}^d$.

One application of teleportation:

Quantum Repeaters

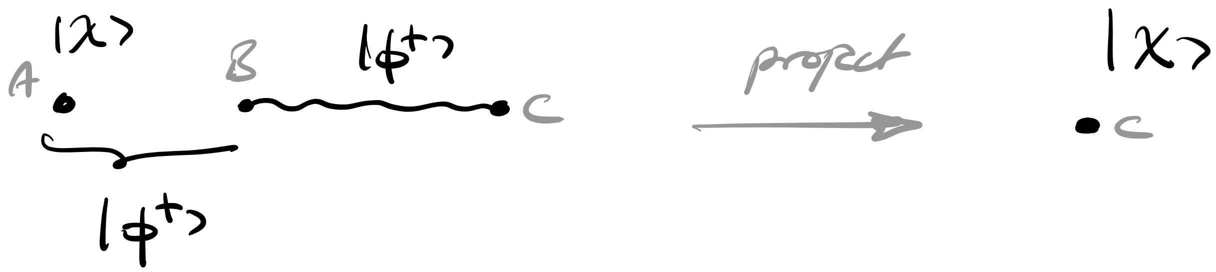
We can (reliably) create entanglement over distance ℓ $\rightarrow$ can we create entanglement over distance 2ℓ?

(E.g.: Photon loss at const. rate $\rightarrow$ prob. to send half of an ent. pair over dist. ℓ is $e^{-\ell/\xi}$.)



b) Relation between teleportation and the Choi-Jamiołkowski isomorphism

① Consider "postselected teleportation"

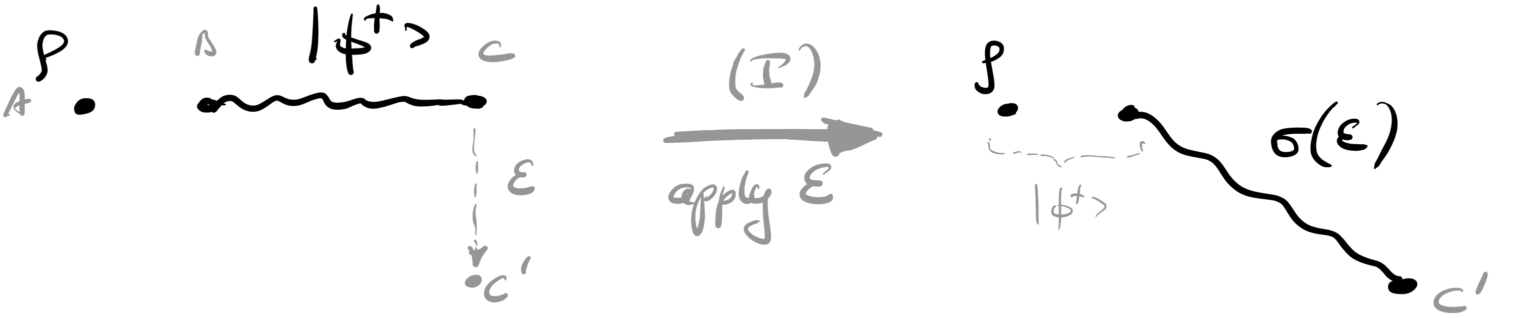


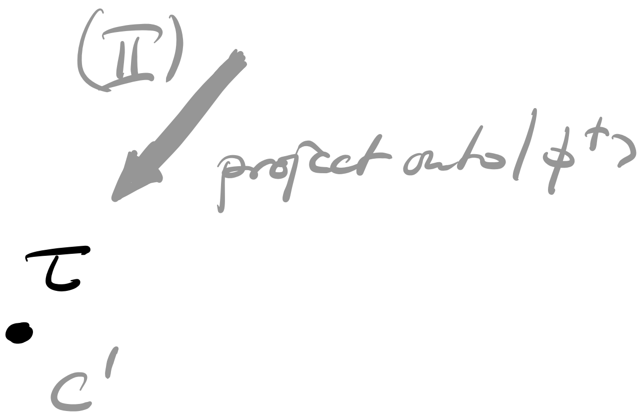
project onto $|\phi^+\rangle$: "postselected" measurement, i.e. we only consider this outcome

... so this is a complicated way of writing the identity map.

② Protocol for applying $\rho \mapsto \mathcal{E}(\rho)$:

③ Now interchange the order of applying $\mathcal{E}$ and projecting — they commute (as they act on diff. systems), so this is the same map:





Since orders commute, this is the same as ②, i.e.

$$\tau = \mathcal{E}(\rho)\,!$$

This is the Choi-Jamiołkowski isomorphism (!):

(I) is the $\mathcal{E} \longmapsto \sigma$ map

("apply $\mathcal{E}$ to half a max. entangled state")

(II) is the $\sigma \longmapsto \mathcal{E}$ map

("teleport σ through the Choi state")

c) Dense coding

Have seen:

- shared entanglement + class. channel $\rightarrow$ q. channel

$$1 \text{ ebit} + 2 \text{ cbit} \longrightarrow 1 \text{ qubit}$$

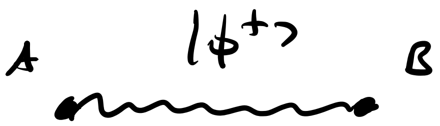
Can we do the converse? Use a quantum channel to transmit classical information?

Trivially possible by encoding $0 \rightarrow |0\rangle$, $1 \rightarrow |1\rangle$

$$1 \text{ qubit} \longrightarrow 1 \text{ cbit}$$

Can we do better if we also share entanglement?

Dense coding (sometimes also "superdense coding"):



Idea: Encode two bits in $\{|\phi_{\alpha\beta}\rangle\}_{\alpha,\beta=0,1}$ (an ONB)

① A & B share $|\phi^+\rangle$.

② A can encode two bits α, β locally:

$$|\phi_{\alpha\beta}\rangle_{AB} = \left(Z_A^{\alpha} X_B^{\beta} \otimes I \right) |\phi^+\rangle_{AB}$$

i.e., A applies $Z^\alpha X^\beta$ to her part of $|\phi^+\rangle$.

③ A sends her part of the state to B via the quantum communication channel.

④ B measures in Bell basis $\{|\phi_{\alpha\beta}\rangle\}$ and recovers α and β.

shared ent. + q. channel $\rightarrow$ class. channel

$$1\,\text{ebit} + 1\,\text{qubit} \longrightarrow 2\,\text{cbit}$$

d) Optimality of teleportation & dense coding

We can use the teleportation & dense coding protocol mutually to argue that both are optimal in terms of communication cost.

To this end, assume shared ent. is free (i.e.: this is not part of our cost function).

i) Assume we can teleport with $r < 2$ bits of class. communication per qubit sent (i.e., there are protocols to send k_q qubits w/ k_c class. bits s.th. $\frac{k_c}{k_q} \longrightarrow r$).

Use this "hyper-teleportation" protocol to send quantum states in the (normal) dense coding prot.:

send $2n$ cbits

$\Big\downarrow$ dense coding

send n qubits

$\Big\downarrow$ "hyper-teleportation"

send rn cbits, $r<2$

$\Rightarrow$ Can compress class. information (in the presence of entanglement).

$\Rightarrow$ Can iterate this to arbitrarily compress class. info – i.e., send n bits with $k \ll n$ bits – as long as we have free entanglement.

This is impossible! (Intuitively, can also be formalized)

ii) Assume we can "hyper-dense-code" $2s > 2$ class. bits per qubit sent.

send $2s$ class. bits

$\Big|$

"hyper-dense coding"

$\downarrow$

send 1 qubit

$\downarrow$ teleportation

send 2 cbits

... and again, we can send 2s bits by only transmitting 2 bits, etc. ...