

## 5. Entanglement of mixed states

Chapter III, pg 70

### a) Introduction

When is a mixed bipartite state  $\rho_{AB}$  entangled?

Different possible definitions:

- (i) If  $\rho_{AB}$  cannot be created by LOCC.
- (ii) If we can extract distill  $| \phi^+ \rangle$  from  $\rho_{AB}$ .
- (iii) If it helps us to do some task better in our LOCC setting ( $\beta$  is a "resource").

... any of those could be in a finite-copy or asymptotic setting!

Clearly, (ii)  $\geq$  (iii)  $\geq$  (i)

↑ " $\geq$ ": is a stronger condition,  
i.e. satisfied by less states.

We use the weakest notion ② to define entangled states.  
Chapter III, pg 71

### Definition:

A bipartite state  $\rho_{AB}$  is called separable if it can be written as

$\cancel{\otimes} \quad \boxed{\rho_{AB} = \sum p_i \rho_i^A \otimes \rho_i^B}$  "separable state"

for some  $p_i \geq 0, \sum p_i = 1;$

$$\rho_i^A \geq 0, \rho_i^B \geq 0; \text{ so } \rho_i^A = \text{tr} \rho_i^B = 1$$

- i.e.,  $\rho_{AB}$  can be prepared by LOCC.

If  $\rho_{AB}$  is not separable - i.e., it has no decomposition of the form  $\otimes$  - it is called entangled.

Given a state  $\rho_{AB}$ , how can we tell if it is entangled?

Problem: Given  $\rho_{AB}$ , need to check all decompositions

$$\rho_{AB} = \sum p_i \rho_i^{AB} = \sum p_i \rho_i^A \otimes \rho_i^B$$

(or  $P_{\text{sep}} = \sum p_i |q_i^{\text{to}} X_i^{\text{to}}|$ ) to find if there is a separable one. (Ambiguity in ensemble decomposition is an isometry, i.e., we need to optimize over 1D vectors!)

→ Difficult!

(In fact, the general problem has been shown to be NP-hard in the dimension of the space.)

→ Need partial solutions, e.g. ways to certify a given state is entangled.

### b) Entanglement witnesses

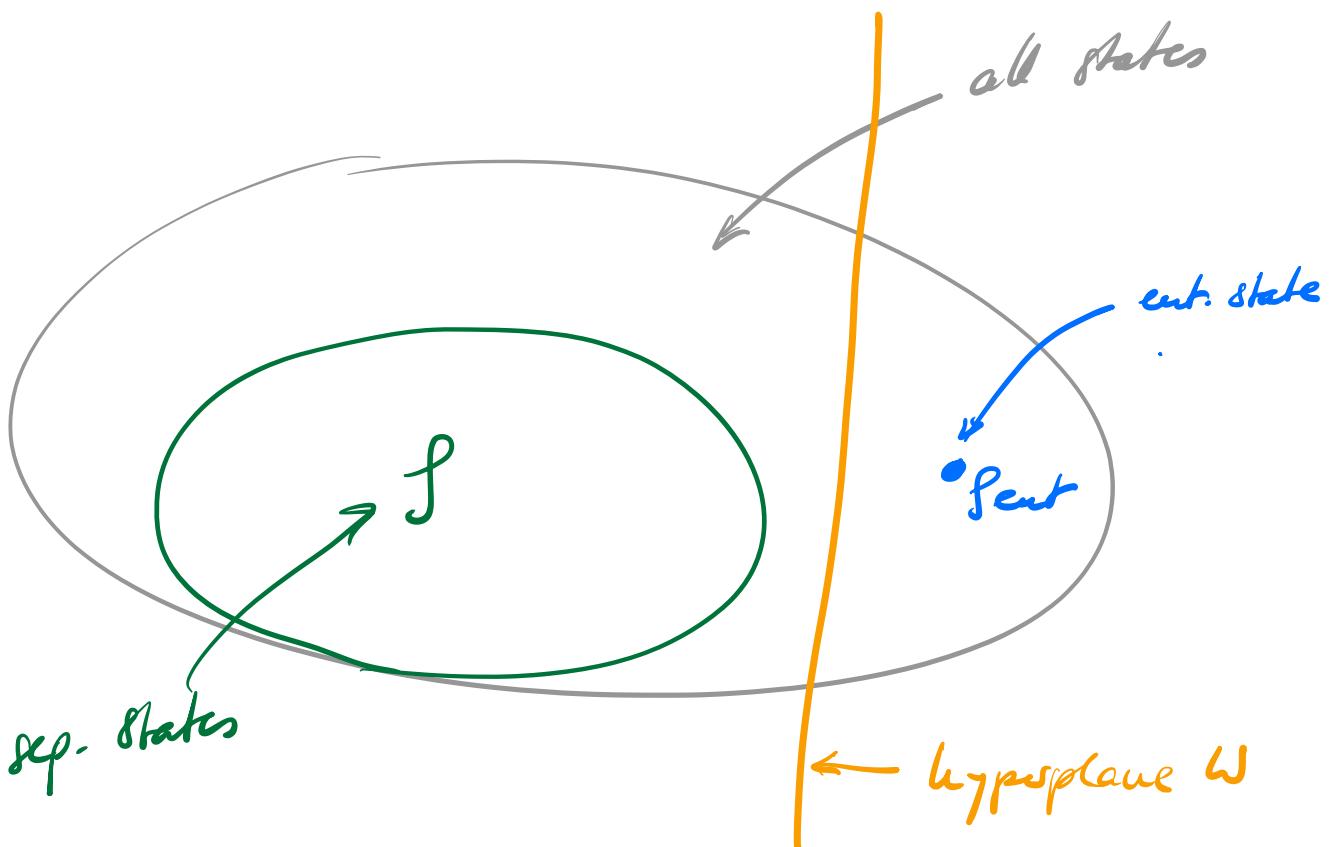
Structure of the set  $\mathcal{S}$  of separable states:

$$\text{Let } \rho = \sum_{i=1}^n p_i \rho_i^A \otimes \rho_i^B \in \mathcal{S}, \quad \sigma = \sum_{j=1}^m q_j \sigma_j^A \otimes \sigma_j^B \in \mathcal{S}$$

$\Rightarrow$  for  $\lambda \in [0; 1]$ ,

$$\underline{\lambda p + (1-\lambda) \sigma} = \lambda p_1 p_1^A \otimes p_1^B + \dots + \lambda p_N p_N^A \otimes p_N^B + \dots \\ + (1-\lambda) q_1 q_1^A \otimes q_1^B + \dots + (1-\lambda) q_M q_M^A \otimes q_M^B \in \underline{P}$$

$\Rightarrow$  The set  $S$  of separable states forms  
a convex set



For any state  $seut \notin S$ , we can find a  
hyperplane which separates  $seut$  from  $S$ .

More generally, we can construct hyperplanes

Chapter III, pg 74

s.t., all points on one side of the plane  
are entangled (but not the other way).

Any hyperplane is of the form  $\text{tr}[X\rho] + c = 0$ ,  
i.e.,  $\text{tr}[\underbrace{(X + cI)\rho}_{=: W}] = \text{tr}[W\rho] = 0$ .

We can choose  $W = W^+$  as we work in the space of hermitian matrices.

Theorem:

$\text{tr}(\rho W) > 0$  :  $\rho$  left of hyperplane  $W$

$\text{tr}(\rho W) < 0$  :  $\rho$  right of hyperplane  $W$

i.e. for a hyperplane as above:

$\rho$  separable  $\Rightarrow \text{tr}(W\rho) \geq 0$

and thus

$\text{tr}(W\rho) < 0 \Rightarrow \rho$  entangled.

Definition (entanglement witness):

An operator  $\omega = \omega^*$  such that

$$\rho \text{ separable} \implies \text{tr}(\omega\rho) \geq 0$$

is called an entanglement witness.

Observation: Given an entanglement witness  $\omega$ ,

$$\text{tr}(\rho\omega) < 0 \implies \rho \text{ entangled.}$$

Notes:

- Key point: Need some way to prove that  $\text{tr}(\omega\rho) \geq 0 \quad \forall \rho \in \mathcal{S}$ !
- Any given witness will only detect some entangled states.
- $\mathcal{S}$  is a convex set  $\implies \mathcal{S}$  is fully specified by all its tangent planes  $\implies$  There exists a witness for any entangled states.

- Wihens are linear operators  $\Rightarrow$  they can be experimentally measured and can (and are) thus being used to identify entanglement in experiments.

### Example:

$$W = F := \sum_{i,j=1}^d |i,j\rangle\langle j,i| \quad (\text{the "flip" or swap})$$

Is it a wihens?

$$\text{Let } \rho_{\text{swap}} = \sum_i p_i \rho_i^A \otimes \rho_i^B:$$

$$\text{tr}(W \rho_{\text{swap}}) = \sum_i p_i \text{tr}(F(\rho_i^A \otimes \rho_i^B)) = \dots$$

Use:

$$\begin{aligned}
 (1) \quad & \underline{\text{tr}[F \cdot (A \otimes B)]} = \sum_{ij} \text{tr}[|i,j\rangle\langle j,i| A \otimes B] \\
 & = \sum_{ij} \langle j,i| (A \otimes B) |i,j\rangle = \sum_j \langle j| A |j\rangle \langle i| B |j\rangle \\
 & = \underline{\text{tr}(AB)} \quad (\text{the "magic formula"},)
 \end{aligned}$$

$$(2) \quad P, Q \geq 0 \implies \text{tr}(PQ) \geq 0$$

Proof:  $P = \sum p_i |\phi_i\rangle\langle\phi_i|$ ,  $p_i \geq 0$ .

$$\begin{aligned} \text{tr}(PQ) &= \sum_i \text{tr}(p_i |\phi_i\rangle\langle\phi_i| Q) \\ &= \sum_i p_i \underbrace{\langle\phi_i| Q |\phi_i\rangle}_{\geq 0} \geq 0. \end{aligned}$$

$$\dots = \sum_{p_i} \underbrace{\text{tr}(p_i^A p_i^B)}_{\geq 0} \geq 0.$$

$\implies \omega$  is entanglement witness!

Which ent. states does  $\omega$  detect? — Note with dominant anti-symmetric component!

$$\text{E.g. } d=2, \quad |\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) :$$

$$F|\psi^-\rangle = -|\psi^-\rangle \implies \langle\psi^-|F|\psi^-\rangle = -1 < 0 !$$

$$\text{while for e.g. } |\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$

$$F|\phi^+\rangle = |\phi^+\rangle \implies \langle\phi^+|F|\phi^+\rangle = +1 > 0 \not\propto$$

What about mixed states?

E.g. for  $d=2$ :

$$\rho = \lambda |\psi^{-}\rangle\langle\psi^{-}| + (1-\lambda) \frac{\mathbb{I}}{4} \quad ; \quad \lambda \in \left[-\frac{1}{3}, 1\right]$$

"Werner state"

$$\begin{aligned} \text{tr}(\mathcal{F}\rho) &= \lambda \langle \psi^- | \mathcal{F} | \psi^- \rangle + \underbrace{\frac{1-\lambda}{4} \text{tr}(\mathbb{I} \cdot \mathcal{F})}_{= d=2} \\ &= -\lambda + \frac{1-\lambda}{2} = \frac{1}{2}(1-3\lambda) \end{aligned}$$

→ for  $\lambda > \frac{1}{3}$ ,  $\rho$  is entangled

Definition: A witness  $W$  is called optimal if there exist  $\rho$  separable such that  $\text{tr}[W\rho] = 0$ .

(i.e., it touches the convex set & cannot be moved closer; otherwise, we could move it a parallel and get a strictly better witness.)

Is  $\omega = F$  optimal?

Yes, e.g.  $\rho = I_d \otimes I_d$

$$\Rightarrow \text{tr}(\rho F) = 0.$$

Other choices: E.g.  $\omega = I - d/d\|X\|_1$ ,

$$(I) = \frac{1}{d} \sum_{i=1}^d |i\rangle\langle i| \quad \rightarrow \underline{\text{Homework!}}$$

c) Positive maps and the PPT criterion

Reminder: A superoperator  $\Lambda : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$  is called positive if  $\rho \geq 0 \Rightarrow \Lambda(\rho) \geq 0$ .

Usually — i.e. for physical maps — we want  $\Lambda$  in addition to be completely positive, i.e.

$$\rho_{AB} \geq 0 \Rightarrow (\Lambda_A \otimes I_B)(\rho_{AB}) \geq 0.$$

But now we will be interested in positive but not completely positive maps!

Why?

Consider  $\rho_{\text{sep}} = \sum p_i \rho_i^A \otimes \rho_i^B$ . Then,

$$\begin{aligned} (\mathbf{1} \otimes \mathbf{I})(\rho_{\text{sep}}) &= \sum p_i \underbrace{\mathbf{1}(\rho_i^A)}_{\geq 0} \otimes \rho_i^B \\ &=: \tilde{\rho}_i^A \geq 0 \quad (\text{1 positive!}) \\ &= \sum_{\geq 0} p_i \underbrace{\tilde{\rho}_i^A}_{\geq 0} \otimes \underbrace{\rho_i^B}_{\geq 0} \geq 0 \end{aligned}$$

and hence:

Theorem: Let  $\mathbf{1}: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$  be a positive map.

Then,  $(\mathbf{1} \otimes \mathbf{I})(\rho_{AB}) \not\succeq 0 \Rightarrow \rho_{AB}$  entangled.  
 ↑  
 not positive semi-definite,  
 i.e. has negative eigenvalues.

Most important example:

$$\mathbf{1}(\rho) = \rho^T \quad (\text{the transpose map})$$

$$(1 \otimes I)(\rho) =: \rho^{T_A} \quad \text{"partial transpose"}$$

(cf. also section II-5)

E.g.:  $|R\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i; i\rangle$

$$\Rightarrow (|R\rangle\langle R|)^{T_A} = \frac{1}{d} \left( \sum |i; i\rangle\langle j; j| \right)^{T_A}$$

$$= \frac{1}{d} \underbrace{\sum |j; i\rangle\langle i; j|}_{= F!}$$

Not positive - anti-sym. states have neg. eigenvalues,

i.e.:  $|A\rangle = \frac{1}{\sqrt{2}} (|ij\rangle - |ji\rangle), \quad i \neq j$ :

$$\langle A | (|R\rangle\langle R|)^{T_A} | A \rangle = \frac{1}{d} (-1) = -\frac{1}{d}.$$

Mixed states:

E.g.  $\rho = \lambda |R\rangle\langle R| + (1-\lambda) \frac{I}{d^2}, \quad \frac{-1}{d^2-1} \leq \lambda \leq 1$ :

"isotropic state"

For instance for  $d=2$ :

$$\rho = \lambda \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} + (1-\lambda) \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1+3\lambda}{4} & \frac{1-\lambda}{4} & \frac{1-\lambda}{4} & \frac{1-\lambda}{4} \\ \frac{1-\lambda}{4} & \frac{1+3\lambda}{4} & \frac{1-\lambda}{4} & \frac{1-\lambda}{4} \\ \frac{1-\lambda}{4} & \frac{1-\lambda}{4} & \frac{1+3\lambda}{4} & \frac{1-\lambda}{4} \\ \frac{1-\lambda}{4} & \frac{1-\lambda}{4} & \frac{1-\lambda}{4} & \frac{1+3\lambda}{4} \end{pmatrix}$$

$$\Rightarrow \rho^{T_A} = \begin{pmatrix} \frac{1+3\lambda}{4} & & & \\ & \boxed{\begin{matrix} \frac{1+3\lambda}{4} & \frac{1-\lambda}{4} \\ \frac{1-\lambda}{4} & \frac{1+3\lambda}{4} \end{matrix}} & & \\ & & \frac{1+3\lambda}{4} & \\ & & & \frac{1+3\lambda}{4} \end{pmatrix}$$

positive iff  $\left(\frac{1-\lambda}{4}\right)^2 \geq \left(\frac{1}{2}\right)^2 \wedge \frac{1+3\lambda}{4} \geq 0$

$$\Leftrightarrow -1 \leq \lambda \leq \frac{1}{3}$$

$$\Rightarrow \rho^{T_A} \geq 0 \text{ iff } \lambda \in \left[ \frac{-1}{d^2-1} ; \frac{1}{3} \right]$$

$\Rightarrow$  For  $\lambda \in \left( \frac{1}{3}, 1 \right]$ ,  $\rho$  is entangled.

Corollary (PPT criteria):

$$\rho^{T_A} \neq 0 \Rightarrow \rho \text{ entangled.}$$

This is called the PPT (positive partial transpose) criterion, or also NPT criterion.

(not positive...).

Note: PPT criterion - and generally positive maps -  
are invariant under local unitaries on  $\mathcal{B}$   $\rightarrow$   
PPT detects all maximally ent. states (and  
in fact all states  $|X\rangle$  w/ full Schmidt rank),  
as  $|X\rangle = (I_A \otimes \Pi_B) |R\rangle$ ,  $\Pi_B$  invertible).

That is: Positive maps are strange than unitaries  
(with formulae like (etc) - but they cannot  
be measured.

In fact:

Recall: The PPT criteria detects all entangled  
states in  $d_A \times d_B = 2 \times 2$  and  $3 \times 2$  dimensions,  
 $\rho$  entangled  $\iff \rho^{T_A} \neq 0$

(Not proven here.)

Counterexamples exist in  $d_A \times d_B = 3 \times 3$  and  $2 \times 4$ ,  
i.e. states  $\rho$  with  $\rho^{T_A} \geq 0$  which are entangled.

Other example of a positive but not CP map:

$$\Lambda(\rho) = \text{tr}(\rho) I - \rho$$

$$(\Lambda \otimes I)(\rho_{AB}) = (I \otimes \text{tr}_A \rho_{AB}) - \rho_{AB}.$$

$$I \otimes \text{tr}_A \rho_{AB} - \rho_{AB} \geq 0 \Rightarrow \text{not entangled}.$$

The "reduction criterion" for entanglement:

$$\underline{I \otimes \text{tr}_A \rho \neq \rho} \quad (\rightarrow \text{Homework}).$$

d) Relation between witnesses and positive maps Chapter III, pg 85

For each witness  $\omega$ , there is a positive map which detects all entangled states the witness detects - and in fact more.

Connection can be understood via Choi-Jamiołkowski isomorphism.

Reminder (Choi-Jam. isomorph.):

CP map  $\mathcal{E}$  on A state  $\sigma$  on AB ( $d_A = d_B$ )

$$\mathcal{E} \longrightarrow \sigma = (\mathcal{E} \otimes I)(I \otimes \rho)$$

$$\mathcal{E}(\rho) = d \text{tr}_B (\sigma (I \otimes \rho^T)) \longleftarrow \sigma$$

Iso-morphism  $\mathcal{E} \leftrightarrow \sigma$  applies also outside of CP maps and  $\sigma \geq 0$ .

Idea: Interpret  $\omega^T$  as the "Choi state" of a map

1. (Since  $\omega \geq 0$  - otherwise  $\text{tr}(\omega\rho) \geq 0$  if! - 1 is not CP.)

$$\text{l.e.: } \Lambda(X) := d \operatorname{tr}_B (\omega^T (I \otimes X^T)) \\ = d \operatorname{tr}_B (\omega (I \otimes X))^T.$$

Theorem for  $\rho \geq 0$ :

$$\langle \phi | \Lambda(\rho)^T | \phi \rangle = d \langle \phi | \operatorname{tr}_B (\omega (I \otimes \rho)) | \phi \rangle$$

$$= d \operatorname{tr} \left[ \underbrace{\omega (| \phi \rangle \langle \phi | \otimes \rho)}_{\text{sep. state!}} \right] \geq 0 \quad \text{↑ } \omega \text{ witness!}$$

$\Rightarrow \Lambda$  is a positive map!

and

$$\operatorname{tr} [\omega (A \otimes B)] = \operatorname{tr}_A [\operatorname{tr}_B (\omega (I \otimes B)) \cdot A]$$

$$= \frac{1}{d} \operatorname{tr}_A [\Lambda(B)^T A] = \frac{1}{d} \sum_{ij} [\Lambda(B)^T]_{ij} A_{ji}$$

$$= \langle \tau | A \otimes \Lambda(B) | \tau \rangle$$

linearity

$$\operatorname{tr} (\omega \rho) = \langle \tau | (I \otimes \Lambda)(\rho) | \tau \rangle$$

That is:

$$\text{tr}(\omega\rho) < 0 \Rightarrow (\mathbf{I} \otimes \Lambda)(\rho) \geq 0$$

$\Rightarrow \Lambda$  detects all states which  $\omega$  detects.

Conversely: When  $\omega$  amounts to the map  $\Lambda$ ,

but checking for a negative eigenvalue only  
along  $|R\rangle$ : would criteria!

Corollary: A state is separable if and only if

I  $(\Lambda \otimes \mathbf{I})(\rho) \geq 0 \quad \forall$  positive maps  $\Lambda$

(since sep. states  $\leftrightarrow$  witness  $\leftrightarrow$  positive maps).

Example:  $\omega = \tilde{F}$ .

$$\Lambda(X) = d \text{tr}_B \left( \tilde{F}(I \otimes X^T) \right) = d \cdot X^T \cdot I = d X^T$$

"magic formula" for partial trace!

$\Rightarrow \omega = \tilde{F}$  corresponds to PPT criteria!

(But: PPT detect all Bell states,  $\tilde{F}$  only the auth. state)

### c) Quantification of mixed state entanglement

How can we quantify entanglement of mixed states?

- i) Entanglement needed to create the state, e.g.  
minimal amount of  $E(\rho) = S(\text{tr}_A \rho_A | \psi \rangle \langle \psi |)$  needed:

$$E_F(\rho) := \min_{\{\rho_i, |\psi_i\rangle\}} \sum p_i E(|\psi_i\rangle)$$

s.t.  $\sum p_i |\psi_i\rangle \langle \psi_i| = \rho$

"entanglement of formation"

or an asymptotic version (cost per copy)

$$E_C(\rho) := \lim_{N \rightarrow \infty} \frac{1}{N} E_F(\rho^{\otimes N})$$

"entanglement cost"

Already  $E_F$  very hard to compute - need to minimize concave function  $S(\rho)$  over convex set

of decompositions  $\rho = \sum p_i \rho_i$  - i.e. ~~Chap III pg 89~~  
 obtained at boundary (hard!)

$$\text{Also, } E_F(\rho \otimes \sigma) \neq E_F(\rho) + E_F(\sigma)$$

(additivity of  $E_F$  - counterex. exist (Hastings))

But: Analytic formula for  $d_A \times d_B = 2 \times 2$  for  $E_F$   
 exists (Wootters), based on so-called "concurvature".

ii) Extractable entanglement:

"Distillable entanglement"  $E_D(\rho)$

$E_D(\rho) = \text{max. asympt. rate } R = \frac{\eta}{N}$  achievable

for LOCC-protocol  $\mathcal{E}_N$ :  $\|\mathcal{E}_N(\rho^{\otimes N}) - |\phi^+ \rangle \langle \phi^+|^N\| \rightarrow 0$   
 as  $N, \eta \rightarrow \infty$ .

↑ suitable dist meas, typ.  
 trace norm.

Even harder to compute:

Asymptotic ( $N, \eta \rightarrow \infty$ ) and any # of LOCC  
 rounds.

(Versions w/ restricted LOCC rounds with, e.g.,  
one-way dist. entanglement, ...)

Observation:  $E_F(\rho) \geq E_C(\rho) \geq E_D(\rho)$

Generally,  $E_C(\rho) \geq E_D(\rho)$ : for most states,  
process is not reversible.

$\Rightarrow$  no unique measure as for pure states!

Example:  $\rho$  PPT, i.e.  $\rho^{T_A} \geq 0$ .

LOCC preserves PPT.  $\Rightarrow$  PPT states are  
undistillable,  $E_D(\rho) = 0$ .

But there exist PPT states with  $E_C(\rho) > 0$ :

"PPT bound entangled states"

(Note: The converse problem -

does  $\rho^{T_A} \geq 0 \Rightarrow E_D(\rho) > 0$  hold?

- is a big open problem - the existence of  
NPT bound entangled states.)

Problem: These might be natural ent. measures,  
but they are essentially impossible to compute.

→ Computable ent. measure desirable!

Wishlist for a good ent. measure:

- LOCC-monotone: Cannot be increased by LOCC  
(probably most relevant!)
- $E(\rho) = 0$  for sep. states  $\rho$  (and only there?)
- additive:  $E(\rho \otimes \sigma) = E(\rho) + E(\sigma)$
- continuous:  $\rho \approx \sigma \Rightarrow E(\rho) \approx E(\sigma)$   
    ( typ. trace norm )
- $E_D \leq E \leq E_C$ .
- coincides with  $E(|\psi\rangle) = S(\log |\psi\rangle)$  on pure states.
- Computable?

(Almost) impossible to get all - LOCC monotonicity  
is probably the most relevant one.

# Negativity - a computable entanglement measure

Found previously:

$\rho^T_A$  has neg. eigenvalues  $\Rightarrow \rho$  entangled.

Use negative eigenvalues to measure entanglement:

$$\begin{aligned} \text{"Negativity"} \quad W(\rho) &= \frac{1}{2} \left( \underbrace{\sum_i |\lambda_i(\rho^T_A)| - 1}_{\text{eigenvalues}} \right) \\ &=: \|\rho^T_A\|_1 : \text{trace norm} \\ &= \frac{1}{2} \left( \|\rho^T_A\|_1 - 1 \right) \end{aligned}$$

$$= - \sum_{\substack{\text{neg. eigenvalues}}} \lambda_i(\rho^T_A)$$

$$\uparrow \quad \text{holds since } \sum \lambda_i(\rho^T_A) = \text{tr}(\rho^T_A) = \text{tr}(\rho) = 1$$

or "log - negativity"

$$E_N(\rho) = \log_2 \|\rho^T_A\|_1$$

Properties:Negativity  $W$ :

- LOCC - monotone
- not additive
- $W(\rho) = 0$  for  $\rho$  separable,  
but also  $W(\rho)$  on PPT ent. states!
- $\neq E(1_4)$  for pure states
- continuous

Log-negativity  $E_N$ :

- not an LOCC monotone (!)
- additive
- $E_N(\rho) = 0$  for  $\rho$  sep, but also on PPT ent. st.
- $\neq E(1_4)$  for pure states
- continuous