IV Quantum Computing and Quantum Algorithms

1. The circuit model

a) Classical computation

Use of classical computers (abstractly):

Solve problems $\equiv$ compute functions

$$f : \{0,1\}^n \longrightarrow \{0,1\}^m$$

$$\underline{x} = (x_1, \dots, x_n) \longmapsto f(x_1, \dots, x_n)$$

The function f depends on the problem we want to solve, $\underline{x}$ encodes the instance of the problem.

E.g.: Problem = multiplication : $(a, b) \longmapsto a \cdot b$

$$\underline{x} = (\underline{x}^1, \underline{x}^2) \longmapsto f(\underline{x}) = \underline{x}^1 \cdot \underline{x}^2$$

encoded in binary

Problem = Factorisation:

$\underline{x}$: integer; $f(\underline{x})$: list of prime factors (suitably encoded)

More precisely:

Each problem is encoded by a family of functions $f \equiv f^{(n)} : \{0,1\}^{m} \to \{0,1\}^{m'}$, with $m = \text{poly}(n)$, $n \in \mathbb{N}$ – one for each input size.

↑ i.e.: m grows at most polynomially with n (technically, $\exists \alpha > 0$ s.th. $\frac{m}{n^{\alpha}} \to 0$).

(Technical point: It must be possible to "construct the functions $f^{(n)}$ systematically and efficiently", see later!)

Which ingredients do we need to compute a general function f?

(i) $f : \{0,1\}^n \longrightarrow \{0,1\}^m$

$$f(\underline{x}) = (f_1(\underline{x}), f_2(\underline{x}), \ldots, f_m(\underline{x}))$$

where $f_k(\underline{x}) : \{0,1\}^n \longrightarrow \{0,1\}$

$\Rightarrow$ can restrict analysis to boolean functions

$$f : \{0,1\}^n \longrightarrow \{0,1\}.$$

(ii) Define $L = \{\underline{y} \mid f(\underline{y}) = 1\} = \{\underline{y}^1, \underline{y}^2, \ldots, \underline{y}^\ell\}$.

Define $\delta_{\underline{y}}(\underline{x}) = \begin{cases} 0 ; & \underline{x} \neq \underline{y} \\ 1 ; & \underline{x} = \underline{y} \end{cases}$ ← bitwise equality!

Then, $f(\underline{x}) = \delta_{\underline{y}^1}(\underline{x}) \vee \delta_{\underline{y}^2}(\underline{x}) \vee \ldots \vee \delta_{\underline{y}^\ell}(\underline{x})$

"$\vee$": logical "or":

(0 ≡ "false", 1 ≡ "true")

$0 \vee 0 = 0$

$0 \vee 1 = 1$

$1 \vee 0 = 1$

$1 \vee 1 = 1$

"$\vee$" is associative:

$$a \vee b \vee c := (a \vee b) \vee c = a \vee (b \vee c)$$

and commutative: $a \vee b = b \vee a$.

(iii) Define <u>Kronecker</u> δ:

$$\delta_y(x) = \begin{cases} 0 : & y \neq x \\ 1 : & y = x \end{cases}$$

Then,

$$\underline{\delta}_{\underline{y}}(\underline{x}) = \delta_{y_1}(x_1) \wedge \delta_{y_2}(x_2) \wedge \dots \wedge \delta_{y_n}(x_n)$$

"$\wedge$": logical "and":

($0 \equiv$ "false",
$1 \equiv$ "true")

$0 \wedge 0 = 0$

$0 \wedge 1 = 0$

$1 \wedge 0 = 0$

$1 \wedge 1 = 1$

"$\wedge$" is <u>associative</u> & <u>commutative</u>;

"$\wedge$" & "$\vee$" are distributive:

$$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c).$$

(In essence, same rules as $\wedge \to \cdot$, $\vee \to +$)

(iv)

$$\delta_y(x) = \begin{cases} x & \text{if } y = 1 \\ \neg x & \text{if } y = 0 \end{cases}$$

↑

Logical "not": $\neg 0 = 1$

$\neg 1 = 0$

Combine (i) – (iv):

Any $f(\underline{x})$ can be constructed from 4 ingredients:

"and", "or", "not" gates,

plus a "copy" gate $x \longmapsto (x, x)$.

This is called a **universal gate set.**

(Note: In fact, already either $\neg(x \wedge y)$ "nand", or $\neg(x \vee y)$ "nor" are universal, together with "copy".)

This gives rise to the

Circuit model of computation:

The functions $f = f^{(n)}$ which we can compute are constructed by concatenating gates from a

simple universal gate set (e.g. and/or/not/copy) sequentially in time (i.e., there are no loops allowed). This gives rise to a circuit for $f^{(n)}$.

The difficulty ("computational hardness") of a problem in the circuit model is measured by the number $K(n)$ of elementary gates needed to compute $f^{(n)}$ ($\hat{=}$ # of time steps).

We often distinguish two qualitatively different regimes:

$K(n) \sim \text{poly}(n)$: efficiently solvable (class P)
easy problem

$K(n) \gg \text{poly}(n)$ — e.g. $K(n) \sim \exp(n^\alpha)$:
hard problem

(Technical note: We must impose that the circuits

used for $f^{(n)}$ are uniform, i.e. they can be generated efficiently – e.g. by a simple n-independent computer program. More formally, $f^{(n)}$ should be generated by a Turing machine.)

Example:

f = Multiplication:

Efficient:

$$\overbrace{10110}^{\ell} \times \overbrace{10011}^{\ell'}$$

$$\begin{array}{r} 10110 \\ 10110 \\ 10110 \\ \hline 110100010 \end{array} \quad \Big\} \ell'$$

$\ell \times \ell'$ additions:

$O(\ell\ell') \sim O(n^2)$ gates.

f: Factorization.

E.g.: Sieve of Eratosthenes:

$\{0,1\}^n \to$ try about $\sqrt{2^n} \sim 2^{n/2}$ cases

$\Longrightarrow$ hard / exp. scaling.

No efficient algorithm known!

Is a typical problem easy or hard?

$f : \{0,1\}^n \longrightarrow \{0,1\}$

# of different f: $2^{(2^n)}$

(2^n: # of inputs; $f(x) = \{0,1$ for each input)

But: there are only $c^{poly(n)}$ circuits of length $poly(n)$!

(c: # of elem. gates)

$\Rightarrow$ As n gets large, most f cannot be computed efficiently (i.e. with $poly(n)$ operations).

Does the computational power depend on the gate set?

NO! By definition, any universal gate set can simulate any other gate set with constant overhead!

Remark: There is a wide range of alternative models of computation, some more and some less realistic:

- CPU
- parallel computers
- "Turing machines" — tape + read/write head
- cellular automata
- ... and lots of exotic models ...

But: All known "reasonable" models of computation can simulate each other with poly(n) overhead $\Rightarrow$ same computational power (in the sense above).

Church-Turing Thesis: All reasonable models of computation have the same computational power.

b) Reversible circuits

For quantum computing – coming soon – we will use the circuit model.

Gates will be replaced by unitaries.

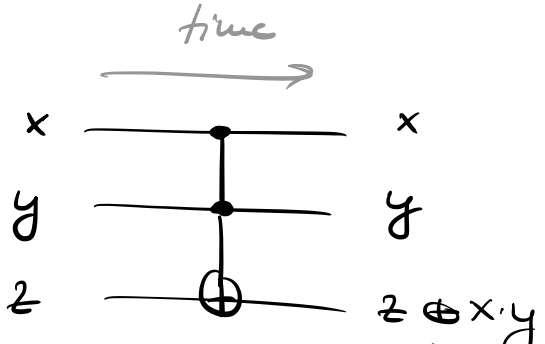
But: Unitaries are reversible,

while classical gates (and, or) are irreversible.

Could such a model even do classical computations – i.e., can we find a universal gate set with only reversible gates?

YES! – Classical computation can be made reversible:

Toffoli gate:



"XOR" = addition mod 2:

$0 \oplus 0 = 0$
$0 \oplus 1 = 1$
$1 \oplus 0 = 1$
$1 \oplus 1 = 0$

also assoc., comm., & distr. w/ $\wedge$. (like "$\vee$")

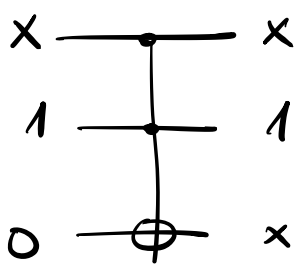
→ Toffoli gate is reversible

(it is its own inverse, since $(z \oplus x \cdot y) \oplus x \cdot y = z$)

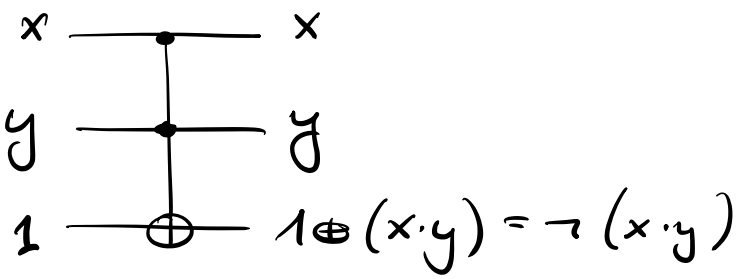
→ Toffoli gate can simulate and/or/not/copy,

by using ancillas in state "0" or "1":

E.g.:



"copy"



"Nand"

⟹ gives reversible universal gate set

(but requires ancillas)

Thus can be used to compute any $f(\underline{x})$ reversibly,

using ancillas, with essentially the same # of gates:

$$f^R(\underline{x}, y) \longmapsto (\underline{x}, f(\underline{x}) \oplus y)$$

↑ bitwise XOR.

(Idea: Replace any gate by a reversible gate using ancillas. Then XOR the result into the y register. Finally, run the circuit backwards to "uncompute" the ancillas. Ancilla count can be optimized for → cf. Preskill's notes.)

⟹ Everything can be computed reversibly.

BUT: 3-bit gate is required! (→ Homework)

c) Quantum Circuits

Most common model for quantum computation:

The circuit model:

- Quantum system consisting of qubits: tensor product structure.
- Universal gate set $S = \{U_1, \dots, U_k\}$ of few-qubit gates (typ. 1- and 2-qubit gates) U_j. (See later for definition of "universal"!)
- Construct circuits by sequentially applying

elements of S to a subset of qubits:

$$|\psi_{out}\rangle = V_T V_{T-1} \cdots V_1 |\psi_{in}\rangle$$

(V_j acting on subset of qubits)

- Initial state:

$$|\psi_{in}\rangle = |x_1\rangle |x_2\rangle \cdots |x_n\rangle \overbrace{|0\rangle|0\rangle \cdots |0\rangle}^{\ell}$$

$$= |\underline{x}\rangle |\underline{0}\rangle$$

($|\underline{x}\rangle$: encodes instance of problem; $|\underline{0}\rangle$: ancillas)

– alternatively, we can also have

$$|\psi_{in}\rangle = |\underline{0}\rangle \equiv |0\rangle^{\otimes \ell}$$

and encode the instance in the circuit.

- At the end of the computation, measure the final state $|\psi_{out}\rangle$ in the computational basis $\{|0\rangle, |1\rangle\}$

⟶ outcome $|y\rangle$ w/ prob. $p(y) = |\langle y|\psi_{out}\rangle|^2$

Notes: • This is a probabilistic scheme — it outputs y w/ some prob. $p(y)$. In principle, we should compare to class. probabilistic schemes — see later.

• We need not measure all qubits —

not measuring = tracing = measuring and ignoring outcome

• POVMs don't help — we can simulate them ($\Rightarrow$ Naimark). Similarly, CP maps don't help — we can simulate them (Stinespring + trace ancilla).

• Measurements at earlier times don't help: Can always postpone them (they commute). If gate at later time would depend on meas. outcome: This dependence can be realized inside the circuit w/ "controlled gates"

(cf. later + homework)

What gate set should we choose?

- There is a continuum of gates — situation much more rich.

- Different notions of universality exist:

  - exact universality: Any n-qubit gate can be realized exactly.

    → Requires a continuous family of universal gates (counting argument!)

  - approximate universality: Any n-qubit gate can be approximated well by gate set

    (Finite gate set sufficient;

    Solovay-Kitaev-Theorem: ε-approximation (in $\|\cdot\|_\infty$-Norm) of 1-qubit gate requires $O(\text{poly}(\log(1/\varepsilon)))$ gates from a suitable finite set.)

- 1- and 2-qubit gates alone are universal!

  (cf. classical: 3-bit gates needed!!)

- For approximate universality, almost any single two-qubit gate will do! (↑ w/ prob. 1.)
- More univ. sets: later!

d) Universal gate set

Our exact universal gate set:

(i) 1-qubit rotations about X & Z axis:

$$R_x(\phi) = e^{-iX\phi/2} \quad ; \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; X^2 = I$$

$$R_z(\phi) = e^{-iZ\phi/2} \quad ; \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; Z^2 = I.$$

For $\Pi^2 = I$: $e^{-i\Pi\phi/2} = \cos\phi/2 \; I - i\sin\phi/2 \; \Pi$

$$\Rightarrow R_x(\phi) = \begin{pmatrix} \cos\phi/2 & -i\sin\phi/2 \\ -i\sin\phi/2 & \cos\phi/2 \end{pmatrix}$$

$$R_z(\phi) = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}$$

Can be understood as rotations on Bloch sphere about X/Z axes by angle ϕ (i.e., rotations in $SO(3) \cong SU(2)/\mathbb{Z}_2$).

Together, R_x and R_z generate all rotations in $SO(3)$ (Euler angles!), and thus in $SU(2)$ up to a phase.

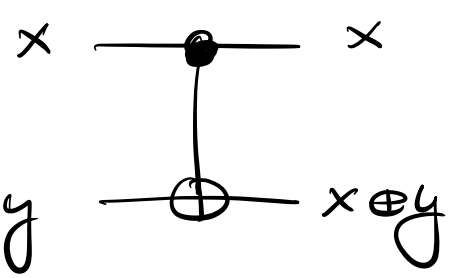
Lemma: For any $U \in SU(2)$,

$$U = e^{i\phi} R_x(\alpha) R_z(\beta) R_x(\gamma) \quad \text{for some } \phi, \alpha, \beta, \gamma.$$

Proof: Homework.

(ii) one two qubit gate (almost all would do!).

Typically, we use "controlled-NOT" = "CNOT":

CNOT = 

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

CNOT flips y iff $x = 1$: classical gate!

Can prove: This gate set can create any n-qubit U exactly (but of course not efficiently - U has $\sim (2^n)^2 = 4^n$ real parameters).

Overview of a number of important gates & identities

(Proof/check: Homework!)

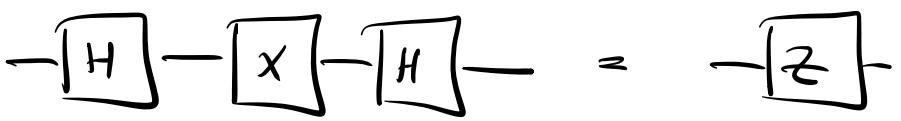
Hadamard gate: $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$H = H^\dagger; \quad H^2 = I.$$

$$H R_x(\phi) H = R_z(\phi)$$

$$H R_z(\phi) H = R_x(\phi)$$

Graphical "circuit" notation:

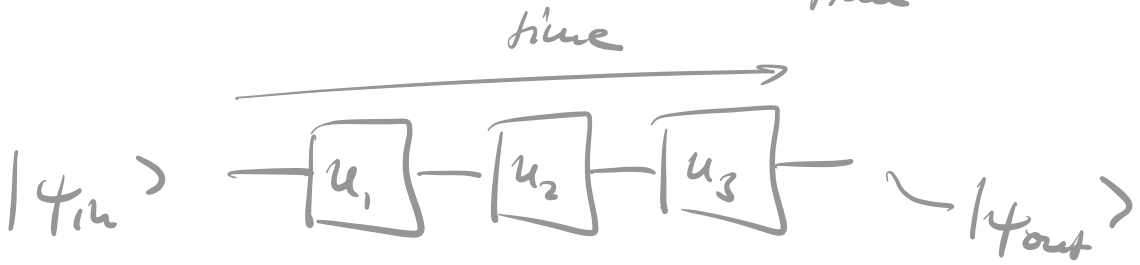


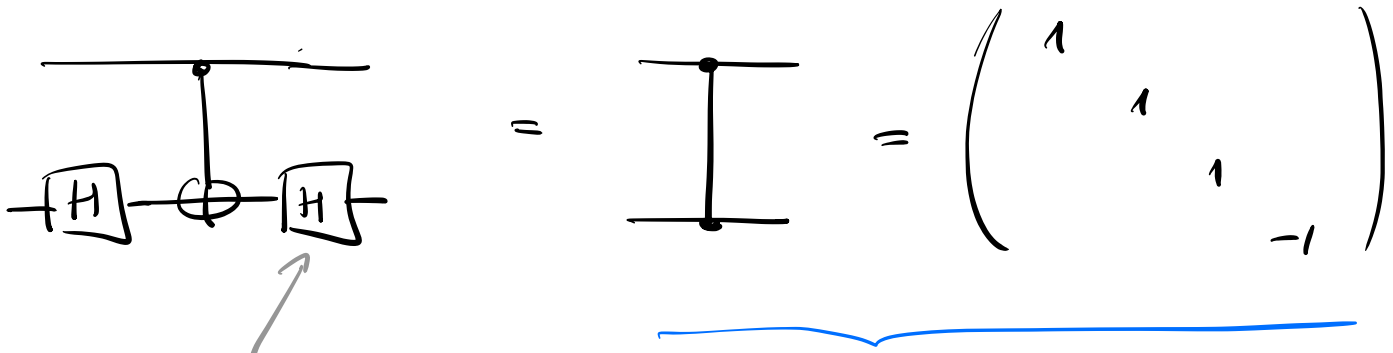
Important:

Matrix notation: time goes right to left

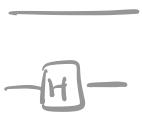
Circuit notation: time goes left to right:

I.e.: $|\psi_{in}\rangle \longmapsto |\psi_{out}\rangle = \underbrace{U_3 U_2 U_1}_{\text{time}} |\psi_{in}\rangle$





only applied to 2nd qubit, i.e.:

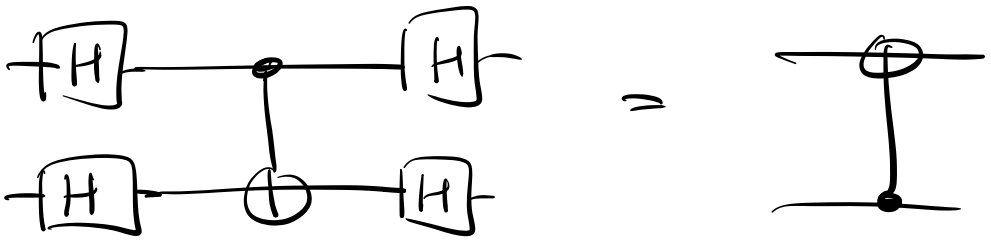


$\equiv I \otimes H$.

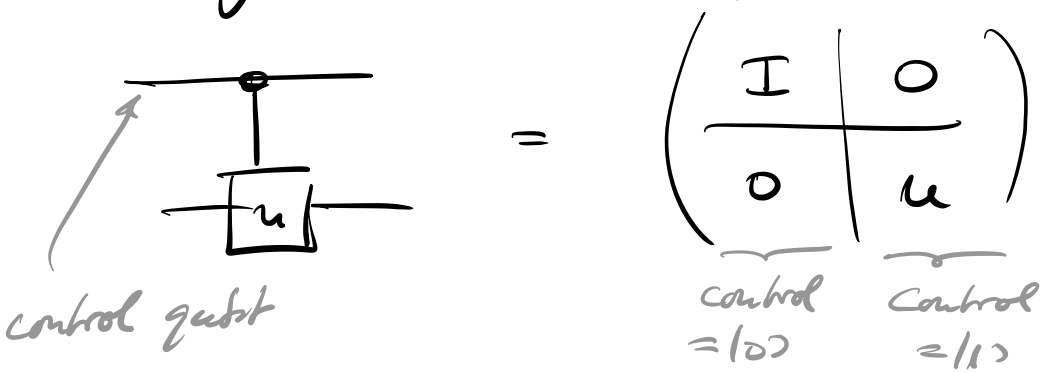
"Controlled-Z"

"Controlled-Phase"

CZ, CPHASE



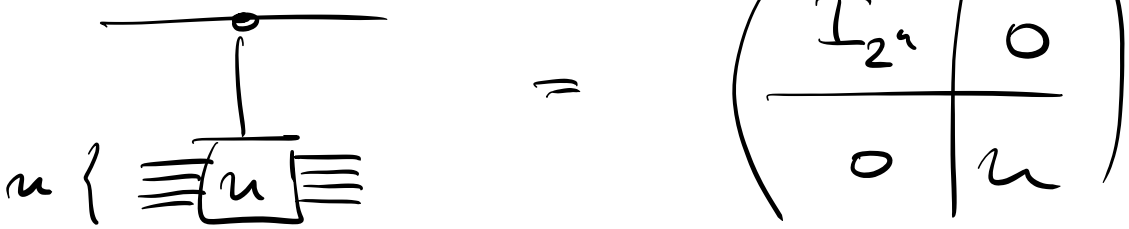
Generally: For a unitary $U \in SU(2)$,



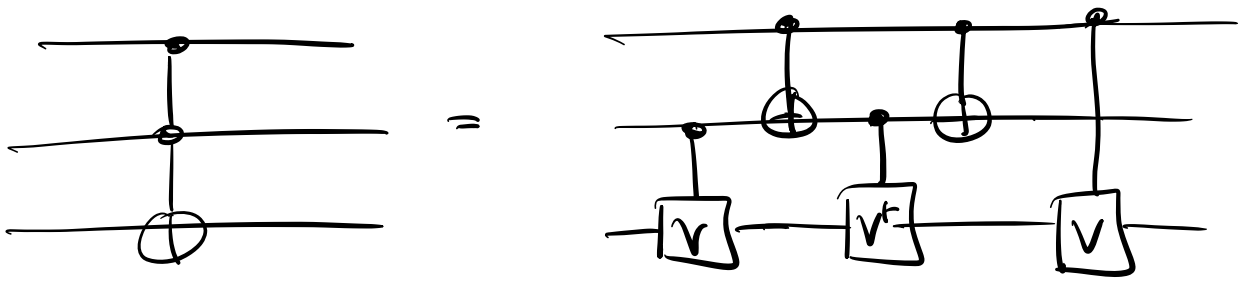
"controlled-U"

Can be implemented w/ 2 CNOT ($\to$ HW!)

Also for $U \in SU(2^n)$:



Circuit for Toffoli:



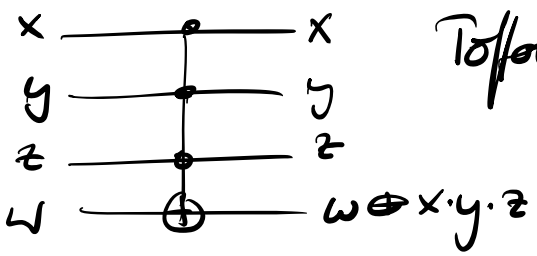
$$\text{with } V = \frac{1-i}{2}(I + iX)$$

U to controlled-U:

Given circuit for U – in particular, a classical reversible circuit — we can also build controlled-U:

Just replace every gate by its controlled-version,

in particular Toffoli by



Toffoli w/ 3 controls can be built from normal Toffoli (since c(G) is universal!)

Finally, some <u>further approx. universal gate sets</u>:

- CNOT + 2 random 1-qubit gates
- CNOT + H + $T = R_z(\pi/4)$ ("$\pi/8$ gate")