

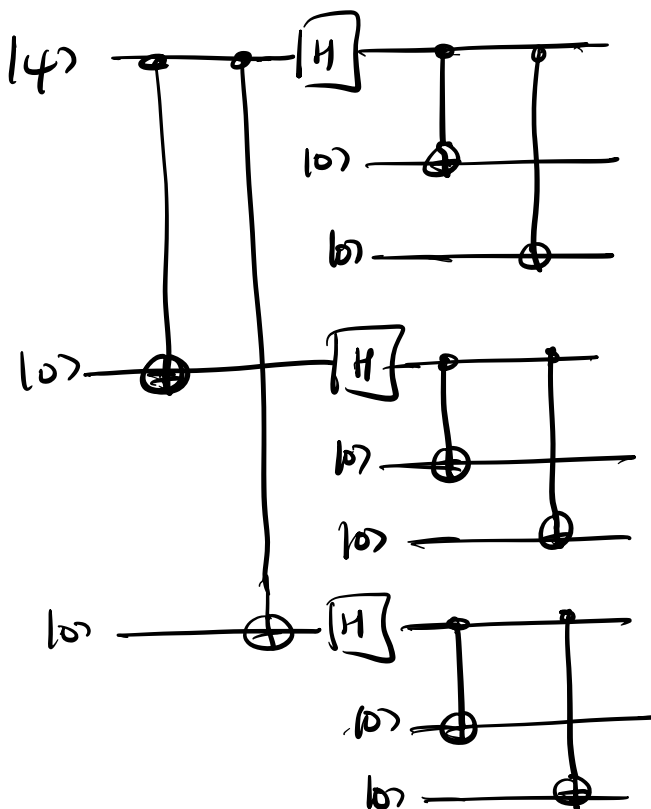
2. The 9-qubit Shor code

Solution: Concatenate (= nest) 3-qubit bit flip code and 3-qubit phase flip code:

$$|0\rangle \mapsto |+\rangle|+\rangle|+\rangle \mapsto \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}}$$

$$|1\rangle \mapsto |-\rangle|-\rangle|-\rangle \mapsto \frac{(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}}$$

Encoding circuit:



9-qubit Shor code

Stor code protects against arbitrary multi-qubit errors! Chapter 7, pg. 14

Sufficient to focus on X , Z , and $Y \propto XZ$ errors:

Any general error $E = \alpha I + \sum \beta_i \sigma_i$ will collapse to one of those (if done right).

— More on this later! —

Intuitively:

(i) Errors X_i are corrected on "inner" layer.

(ii) Z_i error =

= logical error on qubit encoded on inner layer

= Z error on outer layer in one position

$\Rightarrow$ correctable!

(iii) $Y_i \propto X_i Z_i$:

Correct X_i on inner layer.

Then as in (ii): only Z error left!

Note formally: Stabilizers

Code is +1 eigenstate of

$$Z_1 Z_2, Z_2 Z_3 \leftarrow \text{1st row layer}$$

$$Z_4 Z_5, Z_5 Z_6$$

$$Z_7 Z_8, Z_8 Z_9$$

and of

$$X_{(123)} X_{(456)} \leftarrow X \text{ on intermediate qubits}$$

$$\Downarrow \alpha|000\rangle + \beta|111\rangle$$

$$X_{(456)} X_{(789)},$$

$$X_{(123)} = X_1 X_2 X_3$$

$\Rightarrow$

$$X_1 X_2 X_3 X_4 X_5 X_6$$

$$X_4 X_5 X_6 X_7 X_8 X_9$$

These are 8 commuting operators: Measuring
gives 8 bits of information $\Rightarrow$ 1 qubit untouched!

Analysis of errors:Bit flip error X_i :

e.g. X_1 anti-comm. w/ Z_1, Z_2

or X_2 anti-comm. w/ Z_1, Z_2 & Z_2, Z_3

$\Rightarrow$ meas. of all 6 $Z_k Z_l$ reveals position of X_i

$\Rightarrow$ can be corrected!

Phase flip error Z_i :

E.g.: Z_1 : anti-comm. w/ $X_1, X_2, X_3, X_4, X_5, X_6$

But: same holds for Z_2 or Z_3 !

Yet: Z_1, Z_2 and Z_3 act identically on encoded state $|\hat{\psi}\rangle$ - can be seen by inspection, or since

$$Z_2 |\hat{\psi}\rangle = Z_1 \underbrace{(Z_2 Z_1)}_{= |\hat{\psi}\rangle \text{ (stabilizer!)}} |\hat{\psi}\rangle = Z_1 |\hat{\psi}\rangle!$$

(The 9-qubit code is a degenerate code:

different errors have the same syndrome!)

Y errors Y_i :

E.g. $Y_2 \propto Z_2 X_2$

anti-comm. w/ $Z_1 Z_2$

$$Z_2 Z_3$$

$$X_1 X_2 X_3 X_4 X_5 X_6$$

$\Rightarrow$ correctable e.g. via $Z_2 X_2$, or $Z_1 X_2, \dots$

All single-qubit errors can be corrected!

What if errors occur on more than one qubit?

Some - but not all! - can be corrected:

e.g. $X_1 X_4$: correctable.

$Z_1 Z_2$: trivial = no error

but: $X_1 X_2$: breaks inner code $\{$

$z_1 z_4$: breaks outer code $\{$