5. Stabilizer codes

Have seen, e.g. for 3-qubit / 9-qubit code:

code space = joint +1 eigenspace of Paulis

error & correction $\longleftrightarrow$ anti-comm. pattern

$\rightarrow$ General framework?

a) Definition

Definition: The Pauli group $\mathcal{G} \equiv \mathcal{G}_n$ on n qubits is

$$\mathcal{G} := \{ i^\ell P_1 \otimes \ldots \otimes P_n \mid P_i = I, X, Y, Z \,;\, \ell = 0, \ldots, 3 \}$$

Note: Any two $S_1, S_2 \in \mathcal{G}$ either commute or anti-commute.

Definition (Stabilizer group, stabilizer code)

A subgroup $\mathcal{S} \subset \mathcal{G}$ with $-I \notin \mathcal{S}$ is called a stabilizer group $\mathcal{S}$. Since $-I \notin \mathcal{S} \Rightarrow S_1, S_2 \in \mathcal{S}$ commute (else $S_1 S_2 S_1^{-1} S_2^{-1} = -I$); this also implies $S = \pm \otimes P_i \;\; \forall S \in \mathcal{S}$.

The elements $S \in \mathcal{S}$ are called stabilizers.

$\mathcal{S}$ defines a subspace $\mathcal{C} \subset (\mathbb{C}^2)^{\otimes n}$,

$$\mathcal{C} := \{ |\psi\rangle \,|\, |\psi\rangle = S|\psi\rangle \;\; \forall S \in \mathcal{S} \},$$

the code space of a stabilizer code.

$\mathcal{S}$ can also be characterized by a minimal set of generators $S_1, \dots, S_r \in \mathcal{S}$.

Lemma: $\dim \mathcal{C} = 2^{n-r}$.

Proof: (sketch!)

i) S_1 has same # of ± 1 eigenvalues (as $\mathrm{tr}\, S_1 = 0$)

$\Rightarrow$ split space in half.

$\Pi_1 = \frac{1}{2}(I + S_1)$: proj. on $+1$ eigenspace of S_1.

ii) $\Pi_1 S_2 = S_2 \Pi_1$ (as $S_1 S_2 = S_2 S_1$),

and $\Pi_1 S_2 \Pi_1 = \frac{1}{2}(I + S_1) S_2$

± 1 - eigensp. of S_2 on $+1$ - eigenspace of S_1
(0 on -1 - eigensp. of S_1)

$$\mathrm{tr}\left(\frac{1}{2}(I+S_1)S_2\right) = \frac{1}{2}\Big(\underbrace{\mathrm{tr}(S_1)}_{=0} + \underbrace{\mathrm{tr}(S_1 S_2)}_{=0:\ \text{ind. set of gen.-s}}\Big)$$

$$= 0$$

$\Rightarrow$ S_2 has eq. # of $+1/-1$ eigenvals
on $+1$ - eigenspace of S_1
$\Rightarrow$ split again in half.

iii) continue inductively! ☒

b) Error correction conditions for stabilizer codes

What about error corr. conditions?

E_α Pauli errors.

$E_\alpha^\dagger E_\beta$ have three possibilities:

i) $E_\alpha^\dagger E_\beta$ anti-comm. with some $S \in \mathcal{S}$:

$$\langle \hat{i} | E_\alpha^\dagger E_\beta | \hat{j} \rangle = \langle \hat{i} | E_\alpha^\dagger E_\beta \underbrace{S | \hat{j} \rangle}_{S|\hat{j}\rangle = |\hat{j}\rangle}$$

$$= -\langle \hat{i} | S E_\alpha^+ E_\beta | \hat{j} \rangle = -\langle \hat{i} | E_\alpha^+ E_\beta | \hat{j} \rangle$$

$$\Rightarrow \langle \hat{i} | E_\alpha^+ E_\beta | \hat{j} \rangle = 0$$

$\Rightarrow$ QECC satisfied $\Rightarrow$ error correctable!

ii) $E_\alpha^+ E_\beta \in \mathcal{S}$:

$$\langle \hat{i} | \underbrace{E_\alpha^+ E_\beta}_{\in \mathcal{S}} | \hat{j} \rangle = \langle \hat{i} | \hat{j} \rangle = \delta_{ij}$$

$\Rightarrow$ QECC satisfied $\Rightarrow$ error correctable!

iii) $E_\alpha^+ E_\beta$ comm. with all $S \in \mathcal{S}$,

but $E_\alpha^+ E_\beta \notin \mathcal{S}$:

$\Rightarrow$ $E_\alpha^+ E_\beta$ acts non-trivially on code space:

it is a logical operator

In particular: $E_\alpha^+ E_\beta \mathcal{C} \subset \mathcal{C}$, but

$\exists |\hat{j}\rangle$ s.th. $E_\alpha^+ E_\beta |\hat{j}\rangle \neq c \cdot |\hat{j}\rangle$

(else $E_\alpha^+ E_\beta \in \mathcal{S}$)

$\Rightarrow \langle \hat{i} | E_\alpha^\dagger E_\beta | \hat{j} \rangle \neq 0$ for some $i \neq j$

$\Rightarrow$ not correctable! ↯ (as QEC cond. violated)

(Diff. intuition: Cannot tell w/ certainty if after error state is $E_\alpha|\hat{i}\rangle$ or $E_\beta|\hat{j}\rangle \Rightarrow$ not correctable!)

Key question: Given a stabilizer code, what is the shortest $E_\alpha^\dagger E_\beta$ (= Pauli product) of that type (here, "short" refers to # of non-trivial Paulis) ($\rightarrow$ distance of code!)

c) Example: 3-qubit code

$$\mathcal{C} = \text{span}\{|000\rangle, |111\rangle\}$$

$$\left.\begin{aligned} S_1 &= ZZI \\ S_2 &= ZIZ \end{aligned}\right\} \Rightarrow \mathcal{S} = \{III, ZZI, ZIZ, \underset{S_1 S_2}{\overset{}{IZZ}}\}$$

$$k = \underset{n}{3} - \underset{r}{2} = 1 \Rightarrow 1 \text{ encoded qubit}$$

Single-qubit X errors:

$$E_\alpha = III, IIX, IXI, XII$$

$$E_\alpha^\dagger E_\beta = III, IIX, IXI, XII, XXI, XIX, IXX$$

$\Rightarrow$ anti-commu. w/ S_1, S_2, both S_1 & S_2, or an element of S (for III).

$\Rightarrow$ correctable!

Single-qubit Z errors:

$E_\alpha^\dagger E_\beta = ZII$ is one possibility

But: ZII commu. w/ S_1, S_2, but $ZII \notin S$!

$\Rightarrow$ Z errors not correctable!

Logical operators:

(at the same time: uncorrectable $E_\alpha^\dagger E_\beta$!)

- $\hat{Z} = \underbrace{Z I I}_{\text{distance } 1 \,\lightning}$

  - or any $\hat{Z}' = \hat{Z} \cdot S$, $S \in \mathcal{S}$, e.g. IZI, ZZZ, ...

- $\hat{X} = XXX$

  - or e.g. $\hat{X}' = XXX \cdot ZZI = -YYX$, etc ...

<u>Note:</u> $\hat{X}\hat{Z} = -\hat{Z}\hat{X}$ — and this is all we have to require from the logical Pauli operators!

Error detection and correction:

X error E_α can be detected by anti-comm. pattern.

<u>e.g.:</u> • XII anti-comm. w/ ZIZ, $ZZI \in \mathcal{S}$.

• can be measured: $ZZI|\psi\rangle \stackrel{?}{=} \pm |\psi\rangle$ etc.

$\Rightarrow$ allows to detect error (up to a T s.th.

$TS = ST \;\forall S \in \mathcal{S}$, and thus $T \in \mathcal{S}$ for correctable errors).

d) More examples:

3-qubit phase flip code:

$S_1 = XXI$

$S_2 = IXX$

$\hat{X} = XII$

$\hat{Z} = ZZZ$

9-qubit Shor code:

$$\left.\begin{array}{l}
S_1 = ZZI\ III\ III \\
S_2 = IZZ\ III\ III \\
S_3 = III\ ZZI\ III \\
S_4 = III\ IZZ\ III \\
S_5 = III\ III\ ZZI \\
S_6 = III\ III\ IZZ \\
S_7 = XXX\ XXX\ III \\
S_8 = III\ \underline{XXX}\ \underline{XXX}
\end{array}\right\}$$

8 indep. stabilizers

$\parallel$

1 encoded qubit

logical X of 3-qubit code!

Logical operators:

e.g.:

$\hat{Z} = ZZZ\ ZZZ\ ZZZ$

$\hat{X} = XXX\ XXX\ XXX$

— these comm. w/ S_i, as those have even # of X/Z, but are $\notin S$, since they have odd # of X/Z.

simpler ("shorter") logical ops:

e.g. $\hat{Z} = ZII\ ZII\ ZII$

$\hat{X} = XXX\ III\ III$

($\Rightarrow$ distance 3!)

Also means that $\hat{X}$ and $\hat{Z}$ can be measured by measuring only 3 qubits!
(**But**: meas. a general function of $\hat{X}$ & $\hat{Z}$ requires at least 5 qubits because of no-cloning argument!)

<u>Note</u>: 9-qubit code is <u>degenerate</u>:

$E_1 = Z\,I\,I\,I\,I\,I\,I\,I\,I$ and

$E_2 = I\,Z\,I\,I\,I\,I\,I\,I\,I$

have same syndrome, since

$E_1 E_2 = Z\,Z\,I\,I\,I\,I\,I\,I\,I \in \mathcal{S}$.

e) The 5-qubit code

Consider the stabilizer code on 5 qubits w/ generators

$$\left.\begin{aligned} S_1 &= X\,Z\,Z\,X\,I \\ S_2 &= I\,X\,Z\,Z\,X \\ S_3 &= X\,I\,X\,Z\,Z \\ S_4 &= Z\,X\,I\,X\,Z \end{aligned}\right\}$$

encodes $5-4=1$ qubit

<u>cyclic</u> code: $S_1, \dots, S_5$ are cyclic permutations.

$\Rightarrow$ cyclic codewords!

$(S_5 = Z\,Z\,X\,I\,X = S_1 S_2 S_3 S_4)$

<u>Corrects any 1-qubit error</u>:

$E_\alpha^\dagger E_\beta$ = product of ≤ 2 Paulis

$\Rightarrow$ anti-comm. w/ <u>at least</u> one S_i, $i=1,\dots,5$

(Why? Fix pos. of 1st Pauli, pick S_k which has I there. Then, 2nd Pauli must agree with that on S_k; and conversely. But: can check that those choices won't commute w/ some other S_i.)

$\Rightarrow$ correctable $\Rightarrow$ $d \geq 3$.

(And $d \leq 3$ from no-cloning: $[[5,1,3]]$-QECC!)

<u>Error syndromes</u> ($1 \equiv$ anti-comm. = eigenval. -1)

	X error on qubit					Z error on					Y error on				
	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
S_1	0	1	1	0	0	1	0	0	1	0	1	1	1	1	0
S_2	0	0	1	1	0	0	1	0	0	1	0	1	1	1	1
S_3	0	0	0	1	1	1	0	1	0	0	1	0	1	1	1
S_4	1	0	0	0	1	0	1	0	1	0	1	1	0	1	1
S_5	1	1	0	0	0	0	0	1	0	1	1	1	1	0	1

15 errors, 15 syndromes $\Rightarrow$ non-degenerate.

All possible $2^4 - 1 = 15$ syndromes appear.

Logical operators:

$$\hat{Z} = ZZZZZ \qquad \hat{X} = XXXXX$$

} comm. w/ all S_i (even # of X & Z in S_i), but for same reason $\notin \mathcal{S}$!

simpler choices:

e.g. $\hat{Z}' = \hat{Z} \cdot S_3 = -YZYII$

$\hat{X}' = \hat{X} \cdot S_2 = -XIYYI$

$\Rightarrow$ distance $d = 3$

& logical info in $\hat{Z}$ or $\hat{X}$ basis can be obtained by meas. only 3 qubits!

(Note: General feature of distance-d code!)

Syndrome meas. + correction can be done using only CNOT, H, X (for corr.), and ancillas.

(Again: gen. feature of stabilizer code: need to compute parity of X & Z eigenvalues.)

f) Epilogue: Clifford circuits & universal q. computation

Definition (Clifford group)

The Clifford group $\mathcal{C}\ell_n$ on n qubits consists of all gates which map Paulis to Paulis:

$$\mathcal{C}\ell_n = \{ C \text{ unitary} \mid C(P_1 \otimes \dots \otimes P_n)C^\dagger = P_1' \otimes \dots \otimes P_n' \}$$

Theorem (w/out proof):

$$\mathcal{C}\ell_n = \{ \text{all circuits built from CNOT}, S = \begin{pmatrix} 1 & \\ & i \end{pmatrix}, H \}$$

These are also called <u>Clifford gates</u> and <u>Clifford circuits</u>.

(<u>I.e.</u>: Any circuit C which maps Paulis to Paulis is of this form!)

<u>Observation</u>: Only $T = \begin{pmatrix} 1 & \\ & e^{i\pi/4} \end{pmatrix}$ missing for a universal gate set!

Remark: Circuits consisting only of Clifford gates, w/ initial state $|0\rangle^{\otimes n}$, and Pauli measurements, can be simulated efficiently (in essence, by describing state in terms of stabilizers).

$\Rightarrow$ The "simple" gate T gives Clifford circuit universal power!

Key question for error-corrected q. computing:

How can we apply gates on encoded qubits?

Idea I: Decode $\rightarrow$ Apply $\rightarrow$ Encode:

Not good – protection lost during operation!

Idea II: Can we apply gates directly to encoded quantum information?

$\rightarrow$ Focus on stabilizer states/codes!

1. Clifford gates can be applied to encoded qubits:

Clifford gate $\hat{C}$ on logical qubit

$\Updownarrow$

$\hat{C}$ maps logical Paulis to logical Paulis

$\Downarrow$ logical Paulis = products of physical Paulis

$\hat{C}$ maps physical Paulis to physical Paulis

$\Downarrow$

$\hat{C}$ is a Clifford gate on physical qubits.

$\Longrightarrow$ Can implement logical $\hat{C}$ directly on physical qubits.

E.g.: $\hat{H}$ gate on 5-qubit code:

$$\left.\begin{aligned} \hat{X} &= XXXXX \\ \hat{Z} &= ZZZZZ \\ \hat{H}\hat{X}\hat{H} &= \hat{Z} \end{aligned}\right\}$$

need to find Clifford gate s.th. $XXXXX \leftrightarrow ZZZZZ$ & S is preserved.

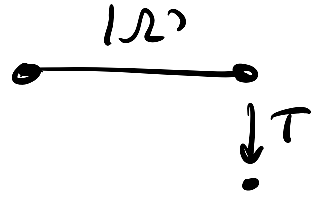
2. Non-Clifford gates:

Can we also realize non-Clifford gates — e.g. $T = \begin{pmatrix} 1 & \\ & e^{i\pi/4} \end{pmatrix}$ — in a robust way?

One idea: "Gate teleportation"

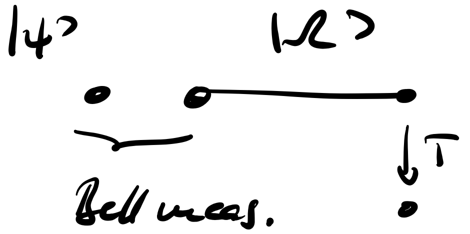
$|\Omega\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

i) Prepare $|\chi\rangle = (I \otimes T)|\Omega\rangle$

$|\chi\rangle =$ 

"magic state"

ii) teleport state $|\psi\rangle$ of q. comp. through $|\chi\rangle$:



↳ state on the right is $T \cdot P|\psi\rangle$, with P a Pauli matrix.

↓

Can be transformed to $T|\psi\rangle$ by Clifford gates!

- Teleportation consists only of Clifford gates and meas. in Z basis $\Rightarrow$ can be done on encoded state.

- Encoded $|X\rangle$ can be prepared before "offline" (e.g. until success).

$\Rightarrow$ Can carry out <u>universal q. computation</u> on <u>encoded qubits.</u>