Lecture & Proseminar 250078/250042
"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2021/22

— Exercise Sheet #1 —

Problem 1: Pauli matrices and Bloch sphere.

1. Check the relation $\sigma_\alpha\sigma_\beta = i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma + \delta_{\alpha\beta}I$ for the Pauli matrices σ_α, $\alpha, \beta, \gamma = 1, 2, 3$.
2. The *trace* $\mathrm{tr}[X]$ is defined as the sum of the diagonal elements of X, i.e., $\mathrm{tr}[X] := \sum_i X_{ii}$. Determine $\mathrm{tr}[I]$, $\mathrm{tr}[\sigma_\alpha]$, and $\mathrm{tr}[\sigma_\alpha\sigma_\beta]$.
3. Determine the eigenstates (=eigenvectors) and eigenvalues of the Pauli matrices.
4. Determine the angles θ and ϕ of those eigenstates on the Bloch sphere, and depict their position on the Bloch sphere.
5. Given a state
$$|\psi\rangle = e^{i\chi}\big[\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle\big] \tag{1}$$
show that
$$|\psi\rangle\langle\psi| = \tfrac{1}{2}(I + \vec{v}\cdot\vec{\sigma}) \quad \text{with } \vec{v}\in\mathbb{R}^3 \text{ and } |\vec{v}| = 1 \ , \tag{2}$$
(i.e., $\vec{v}$ is a vector on the unit sphere in $\mathbb{R}^3$), where $\vec{v}\cdot\vec{\sigma} = \sum_{i=1}^{3} v_i\sigma_i$. (You should find that $\vec{v}$ is exactly the point on the Bloch sphere with spherical coordinates in θ and ϕ, just as introduced in the lecture.)
6. Show that the expectation value of the Pauli operators is $\langle\psi|\sigma_i|\psi\rangle = v_i$; i.e., $|\psi\rangle$ desribes a spin which is polarized along the direction $\vec{v}$.
7. Show that for any state $|\psi\rangle$ with corresponding Bloch vector $\vec{v}$, the state $|\phi\rangle$ orthogonal to it, i.e. with $\langle\psi|\phi\rangle = 0$ (for qubits, i.e., in $\mathbb{C}^2$, this state is uniquely determined up to a phase!), is described by the Bloch vector $-\vec{v}$, i.e., it is located at the opposite point of the Bloch sphere.
(*Bonus question:* Derive a general expression for the overlap $|\langle\phi|\psi\rangle|^2$ of two arbitrary states in terms of the corresponding Bloch vectors.)

(*Note:* A particularly elegant way to check 6. and 7. is to use that $\langle\psi|O|\psi\rangle = \mathrm{tr}[|\psi\rangle\langle\psi|\,O]$ – this is easily shown by writing this explicitly as a sum over components, but you can just this formula as it is if you want, as it will be proven in one of the next lectures – together with Eq. (2) and $\mathrm{tr}[\sigma_i\sigma_j] = 2\delta_{ij}$, but the results can of course also be derived directly from Eq. (1) with a bit more brute force.)

Problem 2: Matrix spaces as Hilbert spaces.

Let $\mathcal{V}_d$ be the space of all complex $d\times d$ matrices, and $\mathcal{W}_d \subset \mathcal{V}_d$ the space of all hermitian complex $d\times d$ matrices (i.e. for $M\in\mathcal{W}_d$, $M = M^\dagger$).

1. Show that $\mathcal{V}_d$ forms a vector space over $\mathbb{C}$, and $\mathcal{W}_d$ forms a vector space over $\mathbb{R}$, but not over $\mathbb{C}$. We will in the following always consider $\mathcal{V}_d$ as a complex and $\mathcal{W}_d$ as a real vector space.
2. Show that the Pauli matrices together with the identity, $\Sigma := \{\sigma_i\}_{i=0}^{3}$, form a basis for both $\mathcal{V}_2$ (over $\mathbb{C}$) and $\mathcal{W}_2$ (over $\mathbb{R}$).
3. Show that
$$(A, B) = \mathrm{tr}[A^\dagger B]$$
defines a scalar product (the "Hilbert-Schmidt scalar product") both for $\mathcal{V}_d$ and for $\mathcal{W}_d$. Here, $\mathrm{tr}[X]$ is the trace, i.e., the sum of the diagonal elements.
4. Show that the Pauli matrices Σ form an orthonormal basis (ONB) with respect to the Hilbert-Schmidt scalar product.

5. Use the fact that for any scalar product $(\vec{v}, \vec{w})$ and a corresponding ONB $\vec{w}_i$, we can write

$$\vec{v} = \sum_i \vec{w}_i(\vec{w}_i, \vec{v}) \ ,$$

to express a general matrix in $M \in \mathcal{V}_2$ as

$$M = \sum m_i \sigma_i \ .$$

What is the form of the m_i? What special property do the m_i satisfy for $M \in \mathcal{W}_2$?

6. Show that a hermitian orthonormal basis also exists for $\mathcal{V}_d$ and $\mathcal{W}_d$. (Ideally, explicitly construct such a basis.)

Problem 3: Unitary invariance and Bell states.

1. Show that the singlet state

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle_{AB} - |10\rangle_{AB})$$

is invariant under joint rotations by the same 2×2 unitary U, i.e.,

$$|\Psi^-\rangle = (U \otimes U)|\Psi^-\rangle$$

for any special unitary matrix $U \in \mathrm{SU}(2)$, i.e. $U^\dagger U = I$, $\det(U) = 1$.

2. Show that this implies that if we measure the spin in any direction $\vec{v}$, $|\vec{v}| = 1$ – this measurement is described by the measurement operator $S_{\vec{v}} = \sum_{i=1}^3 v_i \sigma_i$, i.e. the projectors onto its eigenvectors – we obtain perfectly random and opposite outcomes.

(*Hint:* An elegant way of doing so is to first show that any $S_{\vec{v}}$ has the same eigenvalues as the Z matrix and therefore can be rotated to it, i.e., there exists a $U_{\vec{v}}$ s.th. $U_{\vec{v}} S_{\vec{v}} U_{\vec{v}}^\dagger = Z$. Note that there are very elegant ways to show that the eigenvalues are ± 1 as well!)

3. Determine the states

$$\begin{aligned} &(X \otimes I)|\Psi^-\rangle \ , \qquad (I \otimes X)|\Psi^-\rangle \ , \\ &(Y \otimes I)|\Psi^-\rangle \ , \qquad (I \otimes Y)|\Psi^-\rangle \ , \\ &(Z \otimes I)|\Psi^-\rangle \ , \qquad (I \otimes Z)|\Psi^-\rangle \ . \end{aligned}$$

In the light of point 1, why are they pairwise equal?

Note: Together with $|\Psi^-\rangle$, these are known as the four *Bell states*.

4. Show that the maximally entangled state

$$|\Omega\rangle = \sum_{i=1}^d |i, i\rangle$$

of two qu-d-its (i.e., systems with a Hilbert space $\mathbb{C}^d$) is invariant under $U \otimes \bar{U}$, where U is any $d \times d$ unitary, that is,

$$|\Omega\rangle = (U \otimes \bar{U})|\Omega\rangle \ .$$