

**Problem 9: Measurements and filtering**

Suppose that a bipartite system  $AB$  is initially in the state

$$|\phi_\lambda\rangle = \sqrt{\lambda}|00\rangle + \sqrt{1-\lambda}|11\rangle.$$

The goal of Alice and Bob is to obtain a maximally entangled state

$$|\Omega\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

with some probability by applying local operations only. Specifically, the plan is that Alice will apply a POVM measurement to achieve that.

1. Show that the operators  $M_0 = (|0\rangle\langle 0| + \sqrt{\gamma}|1\rangle\langle 1|)_A \otimes I_B$  and  $M_1 = \sqrt{1-\gamma}|1\rangle\langle 1|_A \otimes I_B$ , with  $0 \leq \gamma \leq 1$ , define a POVM measurement. (Note that these describe measurements carried out on Alice’s side only!)
2. Determine the outcome probabilities and the post-measurement states for both measurement outcomes.
3. Find a value  $\gamma$  such that one of post-measurement states becomes a maximally entangled state. Calculate the corresponding probability with which the initial state becomes a maximally entangled state.
4. In the lecture, we have shown that any POVM measurement can be implemented by adding an auxiliary system in state  $|0\rangle$ , applying a unitary, and measuring the auxiliary system in the computational basis. Construct such a unitary for the POVM of Alice above.

**Problem 10: SIC-POVMs**

A *symmetric informationally complete POVM* (SIC-POVM) in  $d$  dimensions is a POVM  $\{F_i\}_{i=1,\dots,d^2}$  consisting of  $d^2$  operators  $F_i = \lambda\Pi_i$ , where the  $\Pi_i = |\phi_i\rangle\langle\phi_i|$  are rank-1 projectors  $\Pi_i^2 = \Pi_i$ , such that

- i)  $\sum_{i=1}^{d^2} F_i = I$  (i.e. the  $F_i$  form a POVM), and
- ii)  $\text{tr}(F_i F_j) = K$  for  $i \neq j$ , where  $K$  is independent of  $i$  and  $j$  (that is, the POVM is *symmetric*).

1. Use the two conditions (i) and (ii) to determine the values of  $\lambda$  and  $K$ .
2. Now consider  $d = 2$  (qubits). Consider four states  $|\phi_i\rangle$  sitting at the four corners of a tetrahedron. (Any tetrahedron is good, but it might be convenient to have one corner along the  $z$  axis and another one in the  $x$ - $z$ -plane.) Derive the form of  $|\phi_i\rangle$ , and show that they give rise to a SIC-POVM (following the convention above).
3. Show that the operators  $\{F_i\}$  of a SIC-POVM (with the conditions (i) and (ii) above, for arbitrary  $d$ ) are linearly independent. (*Easier version:* Show this only for the qubit SIC-POVM constructed in point 2.)
4. Show that the linear independence of the  $\{F_i\}$  implies that there exist  $G_i$  such that we can write

$$\rho = \sum_{i=1}^{d^2} G_i \text{tr}[F_i \rho]$$

– that is, the POVM is *informationally complete*, i.e., we can reconstruct any state  $\rho$  from the outcome probabilities of the POVM. What is the form of the  $G_i$ ?