Lecture & Proseminar 250078/250042
"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2021/22

— Exercise Sheet #5 —

Problem 11: CHSH inequality I: No-signalling correlations.
Consider the scenario of the CHSH inequality. Let

$$\langle C\rangle = \langle a_0 b_0\rangle + \langle a_1 b_0\rangle + \langle a_0 b_1\rangle - \langle a_1 b_1\rangle \ . \tag{1}$$

Here, $a_x = \pm 1$ and $b_y = \pm 1$ are the outcomes obtained by Alice and Bob given an input (measurement setting) of x (on Alice's side) and y (on Bob's side). As explained in the lecture, the physical theory underlying the CHSH scenario is given by some joint conditional probability distribution $P(a, b|x, y)$ [that is, it has to satisfy $0 \le P(a, b|x, y) \le 1$ and $\sum_{a,b} P(a, b|x, y) = 1$]. The expectation value $\langle a_x b_y\rangle$ is then given by

$$\langle a_x b_y\rangle = \sum_{a,b,x,y} a\, b\, P(a, b|x, y) \ .$$

1. A non-signalling distribution is a distribution which does not allow for communication between Alice and Bob, i.e., Alice's marginal distribution $P^A(a|x) = \sum_b P(a, b|x, y)$ does not depend on Bob's input y, and vice versa. Show that no-signalling distributions can obtain the maximum possible value $|\langle C\rangle| = 4$.
2. Give a distribution $P(a, b|x, y)$ which violates no-signalling.

Problem 12: CHSH inequality II: Tsirelson's bound.
Tsirelson's inequality bounds the largest possible violation of the CHSH inequality (1) in quantum mechanics (namely $2\sqrt{2}$). To this end, let a_0, a_1, b_0, b_1 be Hermitian operators with eigenvalues ± 1, so that

$$a_0^2 = a_1^2 = b_0^2 = b_1^2 = I \ .$$

Here, a_0 and a_1 describe the two measurements of Alice, and b_0 and b_1 those of Bob; in particular, this means that Alice's and Bob's measurements commute, i.e. $[a_x, b_y] = 0$ for all $x, y = 0, 1$. Define

$$C = a_0 b_0 + a_1 b_0 + a_0 b_1 - a_1 b_1 \ .$$

1. Determine C^2.
2. The (operator) norm of a bounded operator M is defined by

   $$\|M\| = \sup_{|\psi\rangle} \frac{\|M|\psi\rangle\|}{\||\psi\rangle\|} \ ,$$

   that is, the norm of M is the maximum eigenvalue of $\sqrt{M^\dagger M}$. Verify that the norm has the properties

   $$\|MN\| \le \|M\|\|N\| \ ,$$
   $$\|M + N\| \le \|M\| + \|N\| \ .$$

   Also, what is the operator norm of a hermitian operator in terms of its eigenvalues?
3. Find an upper bound on the norm $\|C^2\|$.
4. Show that for Hermitian operators $\|C^2\| = \|C\|^2$. Use this to obtain an upper bound on $\|C\|$.
5. Explain how this inequality gives a bound on the maximum possible violation of the CHSH inequality in quantum mechanics. This is known as Tsirelson's bound, or Tsirelson's inequality.