Lecture & Proseminar 250078/250042
"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2021/22

— Exercise Sheet #6 —

Problem 13: Quantum channels.
In this problem, we will study some commonly appearing quantum channels. In addition to the problems listed, verify for each channel that it is a CPTP map (completely positive trace preserving map) and give its Kraus representation.

1. *Dephasing channel.* This channel acts as

$$\mathcal{E}(\rho) = (1-p)\,\rho + p\,Z\rho Z\ .$$

Show that the action of the dephasing channel on the Bloch vector is

$$(r_x, r_y, r_z) \mapsto ((1-2p)r_x, (1-2p)r_y, r_z)\ ,$$

i.e., it preserves the component of the Bloch vector in the Z direction, while shrinking the X and Y component.

2. *Amplitude damping channel.* The amplitude damping channel is giving by the Kraus operators

$$M_0 = \sqrt{\gamma}|0\rangle\langle 1|, \quad M_1 = |0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1|\ ,$$

where $0 \le \gamma \le 1$. Here, M_0 describes a decay from $|1\rangle$ to $|0\rangle$, and γ corresponds to the decay rate.

(a) Consider a single-qubit density operator with the following matrix representation with respect to the computation basis

$$\rho = \begin{pmatrix} 1-p & \eta \\ \eta^* & p \end{pmatrix},$$

where $0 \le p \le 1$ and η is some complex number. Find the matrix representation of this density operator after the action of the amplitude damping channel.

(b) Show that the amplitude damping channel obeys a composition rule. Consider an amplitude damping channel $\mathcal{E}_1$ with parameter γ_1 and consider another amplitude damping channel $\mathcal{E}_2$ with parameter γ_2. Show that the composition of the channels, $\mathcal{E} = \mathcal{E}_1 \circ \mathcal{E}_2$, $\mathcal{E}(\rho) = \mathcal{E}_1(\mathcal{E}_2(\rho))$, is an amplitude damping channel with parameter $1-(1-\gamma_1)(1-\gamma_2)$. Interpret this result in light of the interpretation of the γ's as a decay probability.

3. *Twirling operation.* Twirling is the process of applying a random Pauli operator (including the identity) with equal probability. Explain why this corresponds to the channel

$$\mathcal{E}(\rho) = \tfrac{1}{4}\rho + \tfrac{1}{4}X\rho X + \tfrac{1}{4}Y\rho Y + \tfrac{1}{4}Z\rho Z\ .$$

Show that the output of this channel is the maximally mixed state for any input, $\mathcal{E}(\rho) = \frac{1}{2}I$.

Hint: Represent the density operator as $\rho = \frac{1}{2}(I + r_x X + r_y Y + r_z Z)$ and apply the commutation rules of the Pauli operators.

Problem 14: Gate teleportation.
Gate teleportation is a variation of quantum teleportation that is being used in fault-tolerant quantum computation (a topic which will be covered later in the course of the lecture).
Suppose that we would like to perform a single-qubit gate (i.e., unitary) U on a qubit in state $|\psi\rangle$, but the gate is difficult to perform – e.g., it might fail and thereby destroy the state on which we act on. On the other hand, $U\sigma_j U^\dagger$, where σ_j is any one of the three Pauli matrices, is easy to perform.

1. Verify that such a situation is given when the difficult operation is $U = \left(\begin{smallmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{smallmatrix}\right)$, while Paulis and $S = \left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right)$ are easy to realize.

2. Consider the following protocol to implement U on a state $|\psi\rangle_{A'}$:

   - Prepare $|\chi\rangle_{AB} = (I_A \otimes U_B)|\Phi^+\rangle_{AB}$ (with $|\Phi^+\rangle$ as before. (U_B is still hard, but we can try as many times as we want without breaking anything.)
   - Perform a measurement of $A'A$ in the Bell basis (A' is the register used to store $|\psi\rangle_{A'}$).
   - Depending on the measurement outcome, apply $U\sigma_j U^\dagger$ on the B system.

   Show that this protocol works as it should – that is, it yields the state $U|\psi\rangle$ in the B register with unit probability.

Problem 15: LOCC protocols.
A general LOCC protocol can involve an arbitrary number of rounds of measurement and classical communication. In this problem, we will show that any LOCC protocol can be realized in a single round with only one-way communication, i.e., a protocol involving just the following steps: Alice performs a single measurement described by POVM operators M_j, sends the result j to Bob, and Bob performs a unitary operation U_j on his system.
The idea is to show that the effect of any measurement which Bob can do can be simulated by Alice – in a specific sense, namely up to local unitaries – so all of Bob's actions can be replaced by actions by Alice, except for a final unitary rotation.

1. First, suppose Alice and Bob share the state $|\psi\rangle = \sum \lambda_l |l\rangle_A |l\rangle_B$, and suppose Bob performs a measurement with POVM operators $K_j = \sum_{kl} K_{j,kl} |k\rangle_B \langle l|_B$. Let us denote the post-measurement state by $|\alpha_j\rangle$. On the other hand, suppose that Alice does a measurement with POVM operators with operators $L_j = \sum_{kl} K_{j,kl} |k\rangle_A \langle l|_A$, and denote the post-measurement state by $|\beta_j\rangle$.

   Show that there exist unitaries V_j on system A and W_j on system B such that $|\alpha_j\rangle = (V_j \otimes W_j)|\beta_j\rangle$.

2. Use this to explain how Alice can simulate any POVM measurement of Bob, and how this can be used to implement an arbitrary multi-round protocol with a single POVM measurement $\{M_j\}$ performed by Alice, followed by a unitary operation $\{U_j\}$ on Bob's side by Bob which depends on Alice's outcome.
   (*Hint:* The bases $|l\rangle_A$ and $|l\rangle_B$ above could be an arbitrary orthonormal basis!)