

II. The formulation: States, measurements, and evolution

Chapter II pg 1

1. The formulation of quantum theory

a) Hilbert spaces & bra-ket notation

State of QM system described by vectors in a complex Hilbert space \mathcal{H} . For the purpose of this lecture (and almost all of QM):

\mathcal{H} is a finite dimensional Hilbert space,

i.e. $\mathcal{H} \cong \mathbb{C}^d$.

Ket notation: For a vector in \mathcal{H} , we write

$$|\psi\rangle \in \mathcal{H}.$$

We also call $|\psi\rangle$ a "ket vector" or "ket".

Computational basis: In order to fix isomorphism to \mathbb{C}^d & vector notation, we define a canonical basis, the computational basis

$$|0\rangle, |1\rangle, \dots, |d-1\rangle, \text{ i.e.}$$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots |d-1\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

A general vector is thus of the form

$$|\psi\rangle = v_0|0\rangle + v_1|1\rangle + \dots + v_{d-1}|d-1\rangle$$

$$= \sum_{i=0}^{d-1} v_i|i\rangle = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{d-1} \end{pmatrix}$$

The adjoint vector $(|\psi\rangle)^+$ ← transpose conjugate of the matrix/vector

1's

$$(|\psi\rangle)^+ = (\bar{v}_0, \bar{v}_1, \dots, \bar{v}_{d-1}).$$

We write

$$(|\psi\rangle)^+ =: \langle\psi| \quad \text{"bra vector", "bra"}$$

$$|\psi\rangle = \sum v_i|i\rangle \iff \langle\psi| = \sum \bar{v}_i \langle i|$$

\mathcal{X} is a vector space; we write linear combinations as

$$\lambda|\psi\rangle + \mu|\omega\rangle \in \mathcal{X}.$$

Scalar product:

For two vectors $|\psi\rangle = \sum v_i|i\rangle$, $|\omega\rangle = \sum w_j|j\rangle$, the scalar product is given by

$$(\langle \omega |)^+ \cdot (\langle v |) = \sum \overline{\omega_i} v_i = : \underbrace{\langle \omega | v |}_{\text{"bra-ket"}} :$$

(Note: Sesquilinear in 1st component: $(\lambda \langle \omega |)^+ = \overline{\lambda} \langle \omega |$)

Canonical basis is orthonormal basis (ONB):

$$\langle i | j \rangle = \delta_{ij}.$$

$$\Rightarrow \text{for } |v\rangle = \sum v_i |i\rangle, \quad |\omega\rangle = \sum \omega_j |j\rangle,$$

$$\langle \omega | v \rangle = \sum \overline{\omega_j} v_i \underbrace{\langle j | i \rangle}_{\delta_{ij}} = \sum \overline{\omega_i} v_i.$$

$\| |v\rangle \|_2 := \sqrt{\langle v | v \rangle}$ defines a norm (the 2-norm).

Linear maps:

$H : \mathcal{H} \rightarrow \mathcal{H}$ is a linear map

- with $H |v\rangle := H(\langle v |) -$

$$H(|v\rangle + \lambda |\omega\rangle) = H|v\rangle + \lambda H|\omega\rangle.$$

Note: H always acts on right - i.e.,

$$\langle \omega | H | v \rangle := \langle \omega | (H | v \rangle)$$

The map $I = \sum |i\rangle\langle i|$ satisfies that Chapter II pg 4

$$\text{for } |v\rangle = \sum v_j |j\rangle,$$

$$I|v\rangle = \left(\sum_i (i|X_i) \right) \left(\sum_j v_j |j\rangle \right)$$

$$= \sum_{ij} v_j |i\rangle \underbrace{\langle i|j\rangle}_{\delta_{ij}} = \sum v_j |j\rangle$$

$\Rightarrow I$ is the identity map.

This can also be seen in matrix form:

$$I = \sum_{i=0}^{d-1} |i\rangle\langle i| = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$i \rightarrow \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ $= \begin{pmatrix} 0 & 0 & \cdots & 1 & \cdots & 0 \end{pmatrix}$
 \uparrow

To express a general map M in matrix form,
we can write

$$M = I \cdot M \cdot I$$

$$= \sum_{ij} |i\rangle \underbrace{\langle i|M|j\rangle}_{=: M_{ij} \in \mathbb{C}} |j\rangle$$

$$\begin{aligned}
 &= \sum_{ij} \pi_{ij} \underbrace{|i X j|}_{\parallel} \\
 &\in \left(\begin{array}{ccccccccc} 0 & & & & & & & & 0 \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & \rightarrow & & & & & \\ & & & & 1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 0 & & \\ & & & & & & & \ddots & 0 \end{array} \right) \\
 &= \begin{pmatrix} \pi_{11} & \pi_{12} & \dots & \pi_{1d} \\ \pi_{21} & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots \\ \pi_{d1} & \dots & \dots & \pi_{dd} \end{pmatrix}
 \end{aligned}$$

And similarly for maps $\pi: \mathcal{K}_1 \rightarrow \mathcal{K}_2$.

The map π^+ is the map with entries $\overline{\pi_{ji}}$
 (where $\pi_{ij} = \langle i | \pi | j \rangle$). It holds that
 $(\pi/\omega)^+ = \langle \omega | \pi^+$, and $(AB)^+ = B^+ A^+$.

Unitary maps:

A map $u: \mathcal{K} \rightarrow \mathcal{K}$ is unitary if

$$u^+ u = I,$$

or equivalently:

- $U U^T = I$

- $(U|\omega\rangle)^+ (U|\omega\rangle) = \langle\omega|U^T U|\omega\rangle = \langle\omega|\omega\rangle$

$$\Rightarrow \|U|\omega\rangle\|_2 = \|\omega\|_2$$

(i.e., U preserves angles and areas)

In matrix notation:

$$\langle i|U|j\rangle = U_{ij}$$

$$\underbrace{\langle i|U\left(\sum_k (kXk)\right)U^+|j\rangle}_{= UU^T = I} = \langle i|j\rangle = \delta_{ij}$$

$$\Rightarrow \delta_{ij} = \sum_k U_{ik}(U^T)_{kj} = U_{ik}\bar{U}_{jk}.$$

We call $U = (U_{ij})_{ij}$ a unitary matrix or just a unitary.

Tensor Product:

For $|v\rangle_A \in \mathcal{H}_A \cong \mathbb{C}^{d_A}$, $|\omega\rangle_B \in \mathcal{H}_B \cong \mathbb{C}^{d_B}$,
with comp. bases $\{|i\rangle_A\}_{i=0}^{d_A-1}$, $\{|j\rangle_B\}_{j=0}^{d_B-1}$

$$|v\rangle_A = \sum v_i |i\rangle_A, \quad |\omega\rangle = \omega_j |j\rangle_B,$$

we can define the tensor product

$$|v\rangle_A \otimes |\omega\rangle_B \in \mathcal{H}_A \otimes \mathcal{H}_B = \mathcal{H}_{AB}$$

by defining $\mathcal{H}_A \otimes \mathcal{H}_B$ as the space with ONS
of tuples $(|i\rangle_A, |j\rangle_B)$ with $i=0, \dots, d_A-1$,
 $j=0, \dots, d_B-1$

denoted by

$$|i\rangle_A \otimes |j\rangle_B \quad (\text{or } |i\rangle_A |j\rangle_B, |i,j\rangle_{AB}, |ij\rangle_{AB}, \\ |i\rangle \otimes |j\rangle, |i\rangle |j\rangle, (i,j), |ij\rangle),$$

$$\text{s.t. } (\langle i|_A \otimes \langle j|_B) (|k\rangle_A \otimes |l\rangle_B)$$

$$= \langle i|k\rangle_A \cdot \langle j|l\rangle_B = \delta_{ik} \delta_{jl},$$

and defining $|v\rangle_A \otimes |\omega\rangle_B$ through linearity

$$|v\rangle_A \otimes |\omega\rangle_B = \left(\sum v_i |i\rangle_A \right) \otimes \left(\sum \omega_j |j\rangle_B \right)$$

$$= \left(\sum v_i \omega_j |i\rangle_A \otimes |j\rangle_B \right) - \sum v_i \omega_j |ij\rangle$$

$$= \begin{pmatrix} v_0 \omega_0 \\ v_0 \omega_1 \\ \vdots \\ v_0 \omega_{d_B-1} \\ v_1 \omega_0 \\ v_1 \omega_1 \\ \vdots \end{pmatrix} \quad \begin{matrix} \leftarrow 00 \\ \leftarrow 01 \\ \vdots \\ \leftarrow 10 \end{matrix} \quad \left\{ \begin{array}{l} \text{standard convention} \\ (\text{but } j \text{ rest a conv.!}) \end{array} \right.$$

A general vector $|y\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is of the form

$$|y\rangle = \sum \gamma_{ij} |i\rangle_A |j\rangle_B, \text{ and not necessarily}$$

of the form $|v\rangle \otimes |\omega\rangle$.

Similarly, two maps $M_A : \mathcal{H}_A \rightarrow \mathcal{H}_A$ and $N_B : \mathcal{H}_B \rightarrow \mathcal{H}_B$
always linear!

induce a map

$$(M_A \otimes N_B) : \mathcal{H}_A \otimes \mathcal{H}_B \rightarrow \mathcal{H}_A \otimes \mathcal{H}_B \text{ by virtue of}$$

$$(\Pi_A \otimes N_B)(|v\rangle \otimes |\omega\rangle) := (\Pi_A |v\rangle) \otimes (N_B |\omega\rangle)$$

(and extended linearly to the full space).

In matrix notation,

$$\Pi_A \otimes N_B = \underbrace{\left(\sum |i,j\rangle \langle i,j| \right)}_{\text{res. of identity}} (\Pi_A \otimes N_B) \left(\sum |k,e\rangle \langle k,e| \right)$$

$$= \sum \langle i,j | \Pi_A \otimes N_B | k,e \rangle |i,j \times k,e\rangle$$

$$= \sum \langle i | \Pi_A | k \rangle \langle j | N_B | e \rangle |i,j \times k,e\rangle$$

$$= \sum \underbrace{(\Pi_A)_{ik} (N_B)_{je}}_{\text{ }} |i,j \times k,e\rangle$$

$$= (\Pi_A \otimes N_B)_{(ij), (ke)}$$

$$\Pi_A \otimes N_B = \begin{pmatrix} \Pi_{00} N_{00} & \Pi_{00} N_{01} & \dots \\ \Pi_{00} N_{10} & \dots & \dots \\ \vdots & & & \dots \\ \Pi_{10} N_{00} & \Pi_{10} N_{01} & \dots \\ \Pi_{10} N_{10} & \dots & \dots \end{pmatrix}$$

\approx

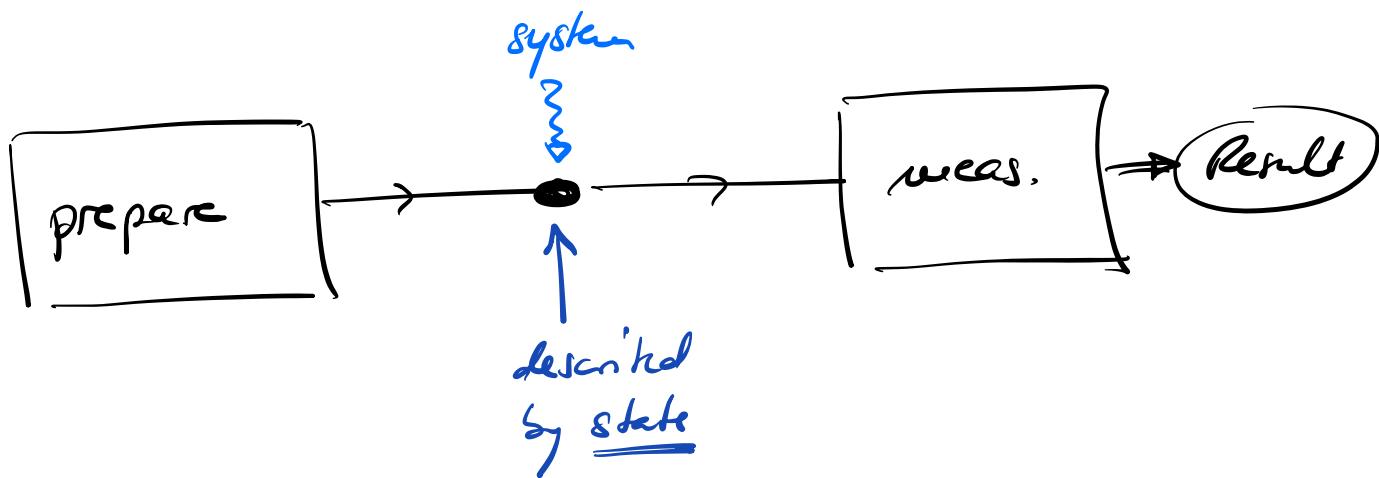
$$\begin{pmatrix} \Pi_{00} \cdot N & \Pi_{01} \cdot N & \dots \\ \hline \Pi_{10} \cdot N & \Pi_{11} \cdot N & \dots \\ \vdots & & \ddots \end{pmatrix}$$

5) The formulation of quantum theory

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Quantum Theory: Framework for theories to describe tests (experiments, games) consisting of preparation and measurement.

(Another theory of this kind is probability theory — we will use it as an analogy, but that's what it is — it sometimes works and sometimes misleads.)



Preparation: Full set of instructions how to prepare system.

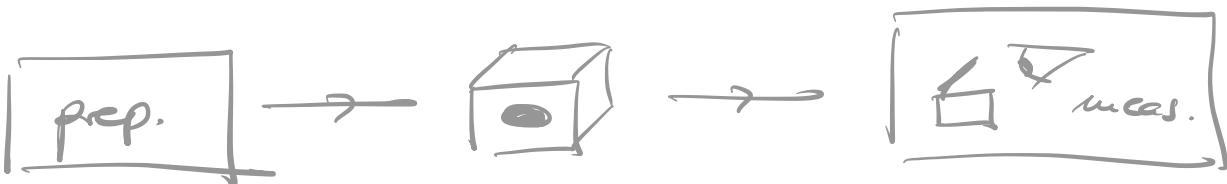
Measurement: Determine some property of phys. system.

Example / Analogy:

- Preparation:
- Put coin in box w/ p_0, p_1 ,
 - Put dice in box w/ p_1, \dots, p_6 .

Requirement: Open box to determine head/tail,
or value of dice.

→ outcome i with prob. p_i .



State: After preparation, we can describe the complete knowledge of the system by assigning a state. The state of the system allows to predict outcomes of measurements as good as possible, given the preparation (could be probabilistic!).

Many different preparation schemes can give identical result for all measurements
 → system described by same state.

I.e.: The state carries all info about preparation relevant for measurement.

Ex: $\vec{p} = \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_n \end{pmatrix}$, or $\vec{p} = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$ is state of coin/dice.

Generally: State in prob. theory is described

by vector $\vec{p} \in \mathbb{R}_{\geq 0}^d$, $\|\vec{p}\|_1 := \sum |p_i| = 1$

Requirement: Outcome i w/ prob.

$$p_i = |\vec{e}_i \cdot \vec{p}|$$

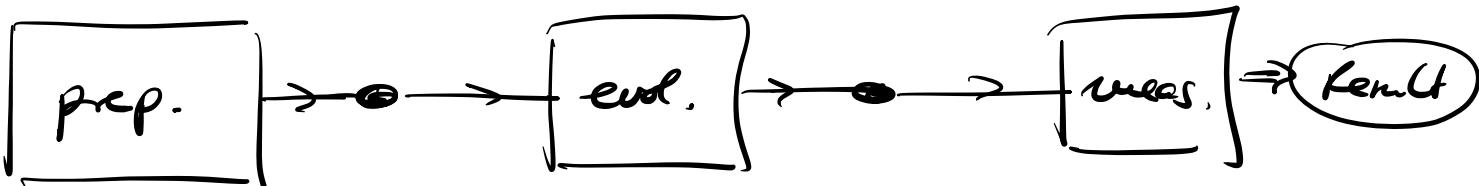
\uparrow i-th unit vector: $\vec{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i$

Collapse: After the measurement, the new state

is $\vec{p}' = \vec{e}_i$: the state collapses onto the outcome.

Note: The state describes our knowledge about the system.

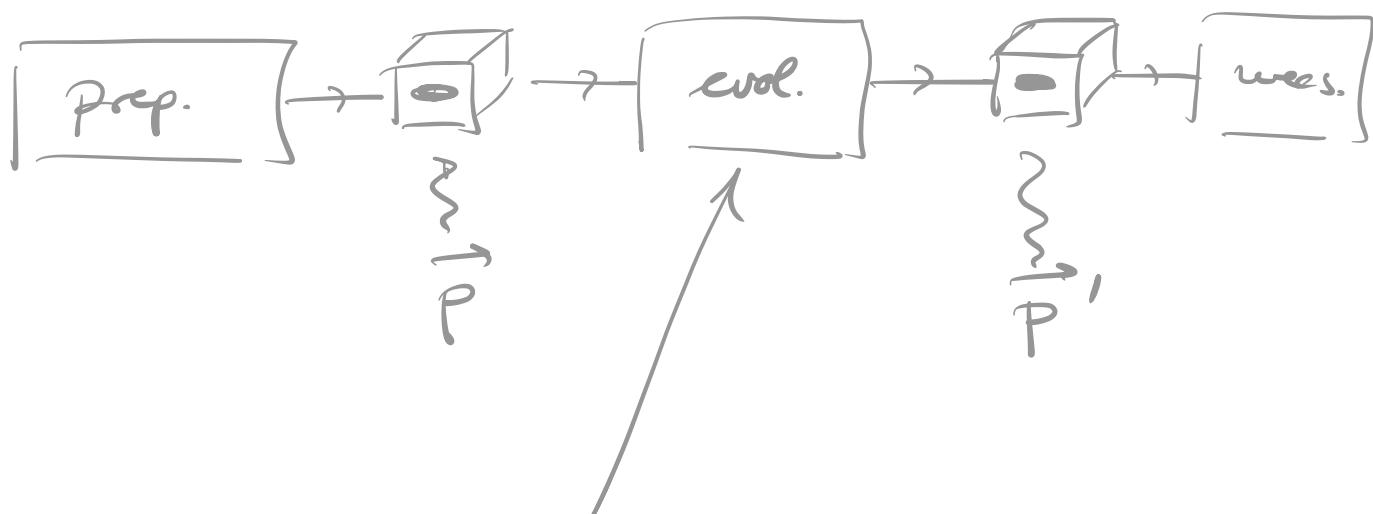
Evolution: In addition, we can "do things" with the system b/w. preparation & measurement,
i.e. evolve it:



Note: evolution can be absorbed into prep. or meas.

- evolution can consist of a sequence of individual evolutions

Ex:



e.g.: • shake box (\rightarrow add randomness)

- put coin heads up
- flip coin / permute dice values
- do one of the above w/ certain probability

Most general evolution:

- 1) decide value of coin/dice/... : i
 - 2) Output j with prob. E_{ji} .
- \rightarrow Need $\sum_j E_{ji} = 1 \quad \forall i$.

i.e. E is a stochastic matrix.

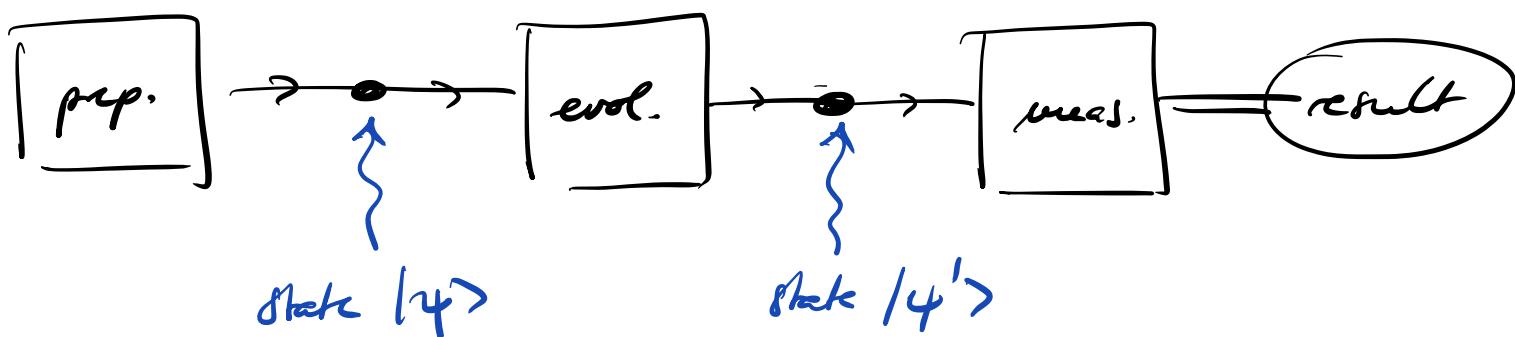
\Rightarrow Evolution maps

$$\vec{p} \mapsto \vec{p}' = E \cdot \vec{p}$$

This is the most general linear evolution such that $\|\vec{p}\|_1 = 1 \Rightarrow \|E\vec{p}\|_1 = 1$!

Quantum Theory:

"Like probability theory, but with the $\|\cdot\|_2$ -norm instead of the $\|\cdot\|_1$ -norm." (\rightarrow Aarnon)



States: $|\psi\rangle \in \mathbb{C}^d \leftarrow$ dim. of H. space;
property of system
 {
 the only property we
 care about - irrespective
 of physical relata.

... such that $\underbrace{\| |\psi\rangle \|_2}_2 = 1$
 often just write $\| |\psi\rangle \|$
 and where $|\psi\rangle$ and $e^{i\phi} \xrightarrow{\phi \in \mathbb{R}} |\psi\rangle$ represent
 the same state.

(i.e. more precisely, states are rays in \mathbb{C}^d , or
 elements of the projective space $\mathbb{C}^d / \mathbb{C}^*$ -
 but we will stick to the convention above.)

Note: State is also often called wavefunction (WF)
 in QM!

State: $|\psi\rangle \in \mathbb{C}^d$, $\| |\psi\rangle \|_2 = 1$, $|\psi\rangle \sim e^{i\phi} |\psi\rangle$

Measurements in Q. Theory:

Let $\{ |b_i\rangle \}$ be an ONS in \mathbb{C}^d , i.e. $\langle b_i | b_j \rangle = \delta_{ij}$.

Then $\{ |b_i\rangle \}$ defines a measurement
 ("measurement on the basis $\{ |b_i\rangle \}$ ")

with the probability p_i of outcome i given by

$$p_i = |\langle b_i | \psi \rangle|^2$$

Note that

$$\begin{aligned}
 \sum p_i &= \sum |\langle \psi_i | \psi \rangle|^2 \\
 &= \sum \langle \psi | b_i \rangle \langle b_i | \psi \rangle \\
 &= \langle \psi | \underbrace{\left(\sum b_i \otimes b_i^\dagger \right)}_{= I} | \psi \rangle \\
 &= \langle \psi | \psi \rangle \\
 &= \| | \psi \rangle \|_2^2 = 1.
 \end{aligned}$$

I.e.: $\| | \psi \rangle \|_2 = 1 \iff$ total probability for some outcome is 1.

Collapse of the state:

After meas. in basis $\{|b_i\rangle\}$ and outcome i ,
the system is described by the state $| \tilde{\psi}_i \rangle = | b_i \rangle$.

\Rightarrow Repeat meas. immediately:

$$p_j' = |\langle b_j | \tilde{\psi}_i \rangle|^2 = \delta_{ij} \Rightarrow \text{same result!}$$

Note: The measurement can also be described through orthogonal projectors $E_i = |b_i\rangle \langle b_i|$. Then, the state $| \tilde{\psi}_i \rangle = E_i | \psi \rangle$ gives us:

- the outcome probability

$$p_i = \|\langle \tilde{\psi}_i \rangle\|^2$$

- the post-measurement state

$$|\psi_i\rangle = \frac{|\tilde{\psi}_i\rangle}{\|\tilde{\psi}_i\rangle\|} = \frac{|\tilde{\psi}_i\rangle}{\sqrt{p_i}}$$

This can be generalized to any complete set of orthogonal projections $\{E_i\}$: $E_i = E_i^\dagger$, $E_i E_j = \delta_{ij} E_i$, $\sum E_i = I$.

Evolution: QM evolution is linear:

$$|\psi\rangle \mapsto U|\psi\rangle$$

It should preserve probabilities, i.e. the total prob. for some outcome sums to 1.

We then require

$$\|U|\psi\rangle\|_2 = \|\psi\rangle\|_2 = 1$$

i.e. U is norm-preserving.

$\Rightarrow U$ is unitary, $UU^\dagger = U^\dagger U = I$.

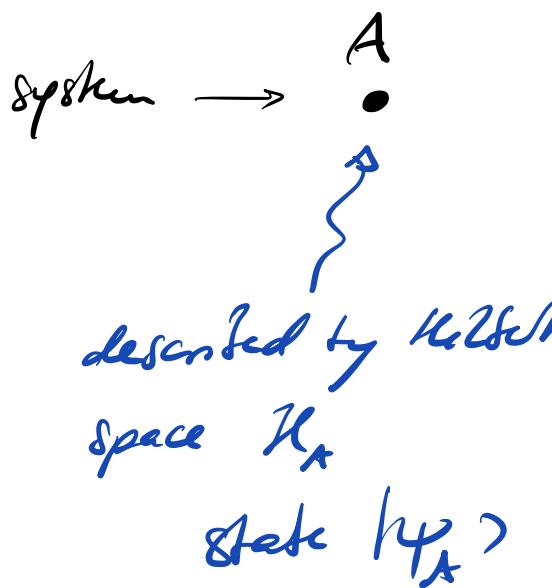
And: Any unitary U is an allowed evolution.

Evolution: $| \psi \rangle \mapsto U | \psi \rangle$, $UU^\dagger = U^\dagger U = I$

Composite systems:

What if we have two parties A & B, who each control a quantum system ("subsystem")?

How should we describe their state?



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A & B should be able to describe their respective system rules. of the other party (\hookrightarrow the rest of the world) \rightarrow states $|\psi_A\rangle, |\psi_B\rangle$.

\rightarrow Joint state of A&B described by

$$|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \in \mathcal{H}_{AB}.$$

What if Alice performs a measurement (given by $\{E_i^A\}$) or evolution (given by U_A)? (Write X_A for E_i^A .)

\rightarrow should be independent of Bob's actions (or even existence).

\rightarrow Action on $|\psi_{AB}\rangle$ given by

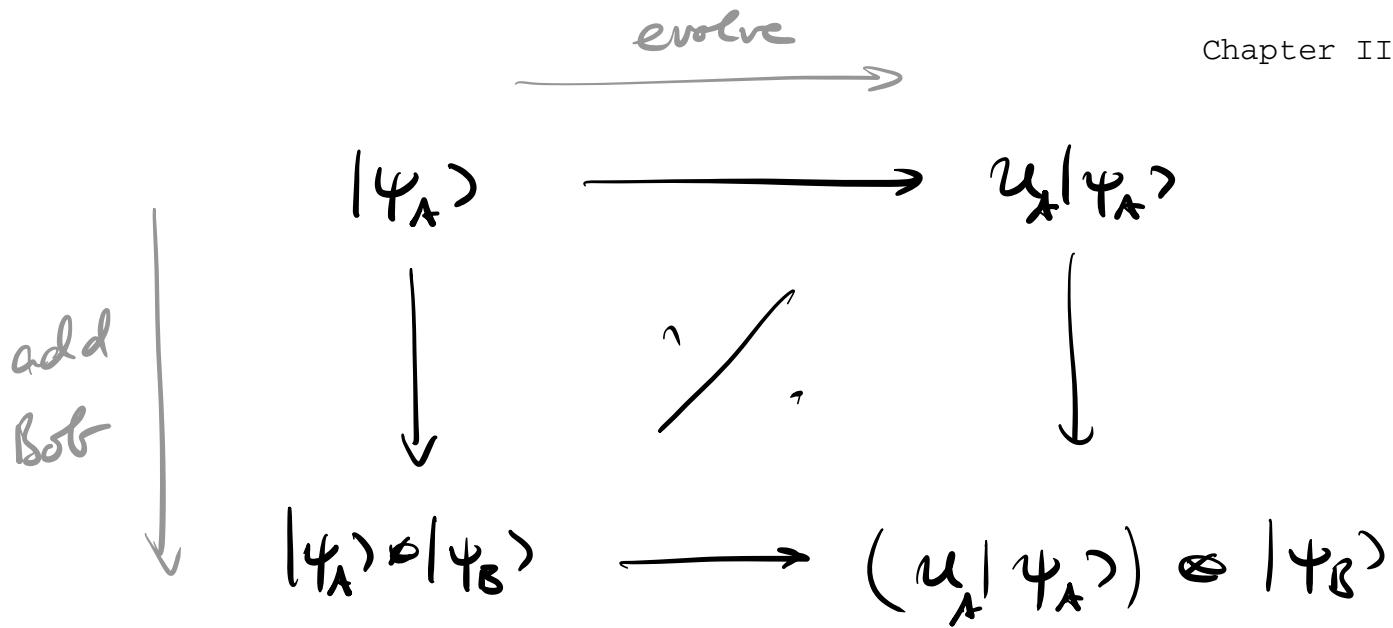
$$|\psi_{AB}\rangle \xrightarrow{\quad} (X_A \otimes I_B) |\psi_{AB}\rangle.$$

||

$$|\psi_A\rangle \otimes |\psi_B\rangle \xrightarrow{\quad} (X_A |\psi_A\rangle) \otimes |\psi_B\rangle + |\psi_B\rangle.$$

Why is this a good (correct) choice?

- E.g.: Alice evolves her state with U_A :



- or; measure $\{E_i^A\}$. Prob.:

$$\| (E_i^A \otimes I) |\psi_A\rangle \otimes |\psi_B\rangle \|_{}^2 =$$

$$= (\langle \psi_A | \otimes \langle \psi_B |) (E_i^A \otimes I) |\psi_A\rangle \otimes |\psi_B\rangle$$

$E_i^A = E_i$

$$= \underbrace{\langle \psi_A | E_i^A | \psi_A \rangle}_{\| E_i^A |\psi_A \rangle \|_{}^2} \cdot \underbrace{\langle \psi_B | \psi_B \rangle}_{= 1}$$

Note: If both A & B act with $X_A \otimes Y_B$, then chapter II pg 22

$$\text{total action is } (I \otimes Y_B)(X_A \otimes I) = X_A \otimes Y_B.$$

Notes: • By linearity, this can be extended to all states on $\mathcal{H}_A \otimes \mathcal{H}_B$ (i.e. not of the form $| \psi_A \rangle \otimes | \psi_B \rangle$).

- The post-measurement state of a measurement $\{E_i^A\} = \{E_i^A \otimes I_B\}$ is

$$| \psi_i \rangle \propto (E_i^A \otimes I_B) | \psi \rangle.$$

- Works the same for composition of more systems (e.g. molecules!)

Analogy - probability:

2 coins with $\vec{P}_A = \left(\frac{1}{3}, \frac{2}{3} \right)$, $\vec{P}_B = \left(\frac{1}{4}, \frac{3}{4} \right)$

\Rightarrow total prob. dist. has 4 poss'ls: 00, 01, 10, 11

with

$$\underbrace{(P_{00}, P_{01}, P_{10}, P_{11})}_{\vec{P}_{AB}} = \left(\frac{1}{3} \cdot \frac{1}{4}, \frac{1}{3} \cdot \frac{3}{4}, \frac{2}{3} \cdot \frac{1}{4}, \frac{2}{3} \cdot \frac{3}{4} \right) = \vec{P}_A \otimes \vec{P}_B.$$

Flipping the first coin — i.e. $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ — acts on Chapter II, pg 23

$$\xrightarrow{\text{Prob}} \text{as } X \otimes I = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix}.$$

Measuring the value of the 1st coin — given by projections $E_0^A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_1^A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, amounts on

$$\xrightarrow{\text{Prob}} \text{to } E_0 = E_0^A \otimes I = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 0 \\ & & 0 & 0 \end{pmatrix} \dots \text{etc.}$$

Quantum mechanical actions (pure state version):

- Systems are described by Hilbert spaces $\mathcal{H} \cong \mathbb{C}^d$.
- States are normalized vectors $|q\rangle \in \mathcal{H}, \| |q\rangle \| = 1, |q\rangle \sim e^{i\phi} |q\rangle$
- Evolutions $|q\rangle \mapsto U|q\rangle$ are unitary, $U^\dagger U = UU^\dagger = I$
- Measurements are given by complete sets of orth. projectors $\{E_i\}$, $E_i = E_i^+$, $E_i E_j = \delta_{ij} E_i$,

$\sum E_i = I$, by virtue of

$$|\tilde{\psi}_i\rangle := E_i |\psi\rangle$$

$$\text{with prob. } p_i = \|\tilde{\psi}_i\|_2^2 = \langle \tilde{\psi}_i | \tilde{\psi}_i \rangle$$

$$\text{and post-meas. state } |\psi_i\rangle = \frac{|\tilde{\psi}_i\rangle}{\|\tilde{\psi}_i\|}$$

- Composite systems are described by tensor products $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. Independent states $|\psi_A\rangle, |\psi_B\rangle$ give a state $|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \in \mathcal{H}_{AB}$, and indep. operations (evol., meas.) X_A, Y_B act as $X_A \otimes Y_B$ (where "doing nothing" = I).

Notes: • In "traditional" physics teaching, measurements are described by hermitian "observables" $O = O^\dagger$, where the measurement yields an "expectation value" $\langle \psi | O | \psi \rangle$.

If we write O in its spectral decomposition,

$$O = \sum \lambda_i E_i \quad \text{non-degenerate!}$$

then $\langle \psi | O | \psi \rangle = \sum \lambda_i \langle \psi | E_i | \psi \rangle =$

$$= \sum p_i \lambda_i$$

- i.e., outcome i has the value λ_i assigned, and we measure the average value (and weaker echo of a measurement).

In Quantum Information, when we say "we measure O ", we in fact mean "we measure $\{E_i\}$ ".

• In physics, evolutions are generated by a Hamiltonian, i.e. by a hermitian operator $H=H^\dagger$, by virtue of

$$U = \exp(-iHt),$$

where t is time (i.e., evolutions are continuous!)

$$\left(\rightarrow \text{Schrödinger equation} \quad \frac{d}{dt} |\psi\rangle = -iH |\psi\rangle \right)$$

c) Examples:

Qubit $\mathcal{H} = \mathbb{C}^2$;

"computational basis" $\{|0\rangle, |1\rangle\}$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1$$

Measurement in basis $\{|0\rangle, |1\rangle\}$, i.e.

$$E_0 = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_1 = |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(Corresponds e.g. to observable $\mathcal{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$)

Measurement:

$$\text{Outcome 0: } |\tilde{\psi}_0\rangle = E_0|\psi\rangle = \alpha|0\rangle$$

$$\rightarrow \text{prob. } p_0 = \|\alpha|0\rangle\|^2 = |\alpha|^2$$

$$= \langle \psi | E_0 | \psi \rangle = |\alpha|^2$$

$$= |\langle 0 | \psi \rangle|^2 = |\alpha|^2$$

$$\text{Post-meas. state } |\tilde{\psi}_0\rangle = \frac{|\tilde{\psi}_0\rangle}{\|\tilde{\psi}_0\|} = |0\rangle$$

$$\underline{\text{Outcome 1}} : |\tilde{\psi}_1\rangle = E_1 |\psi\rangle = \beta |1\rangle$$

$$P_1 = \| |\tilde{\psi}_1\rangle \|^2 = |\beta|^2$$

$$|\psi_1\rangle = |1\rangle$$

Requirement "in X basis", i.e. of observable

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |+X+| - |-X-|, \text{ with}$$

$$|+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) ; \quad |- \rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle).$$

$$|\tilde{\psi}_{\pm}\rangle = |+X_{\pm}| \psi\rangle = |+ \rangle \left(\frac{1}{\sqrt{2}} (\alpha |0\rangle + \beta |1\rangle) \right)$$

$$= |+ \rangle \left(\frac{1}{\sqrt{2}} (\alpha \pm \beta) \right)$$

$$\Rightarrow P_{\pm} = \frac{1}{2} |\alpha \pm \beta|^2 \quad \leftarrow \text{prob.}$$

$$|\psi_{\pm}\rangle = |\pm\rangle \quad \leftarrow \text{post-meas. state}$$

Note: Outcomes can also be labelled by eigenvalues,
e.g. outcomes +1 and -1 for $X \& Z$.

Important: Pauli matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- often also written $\sigma_x = X, \sigma_y = Y, \sigma_z = Z$,
 or $\sigma_1 = X, \sigma_2 = Y, \sigma_3 = Z$. Sometimes
 also $\sigma_0 = I$.

- satisfy $XY = iZ$ & cyclic: $YZ = iX$
 $ZX = iY$
- def. Pauli's anti-comm: $XY = -YX$ etc
- in addition $X^2 = Y^2 = Z^2 = I$
- summarized as $\sigma_x \sigma_y = i \sum_{\alpha, \beta} \epsilon_{\alpha \beta \gamma} \sigma_{\gamma} + \delta_{\alpha \beta} I$
 \uparrow
 fully anti-symmetric tensor.

The Pauli matrices are hermitian and
 unitary, i.e. can describe both measurements
and evolution!

Evolutor:

Consider $U = H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ "Hadamard gate"

$$U|q\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} (\alpha|0\rangle + \beta|1\rangle)$$

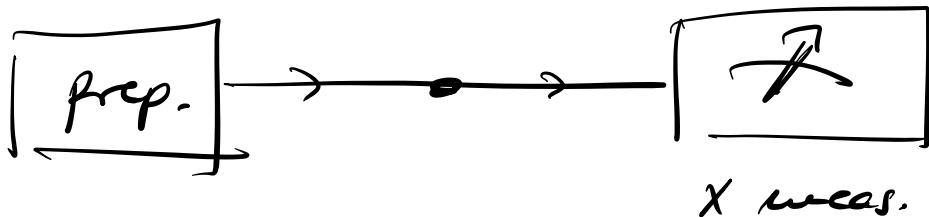
$$= \frac{\alpha + \beta}{\sqrt{2}} |0\rangle + \frac{\alpha - \beta}{\sqrt{2}} |1\rangle$$

Requirement in 2-basis $\{|0\rangle, |1\rangle\}$:

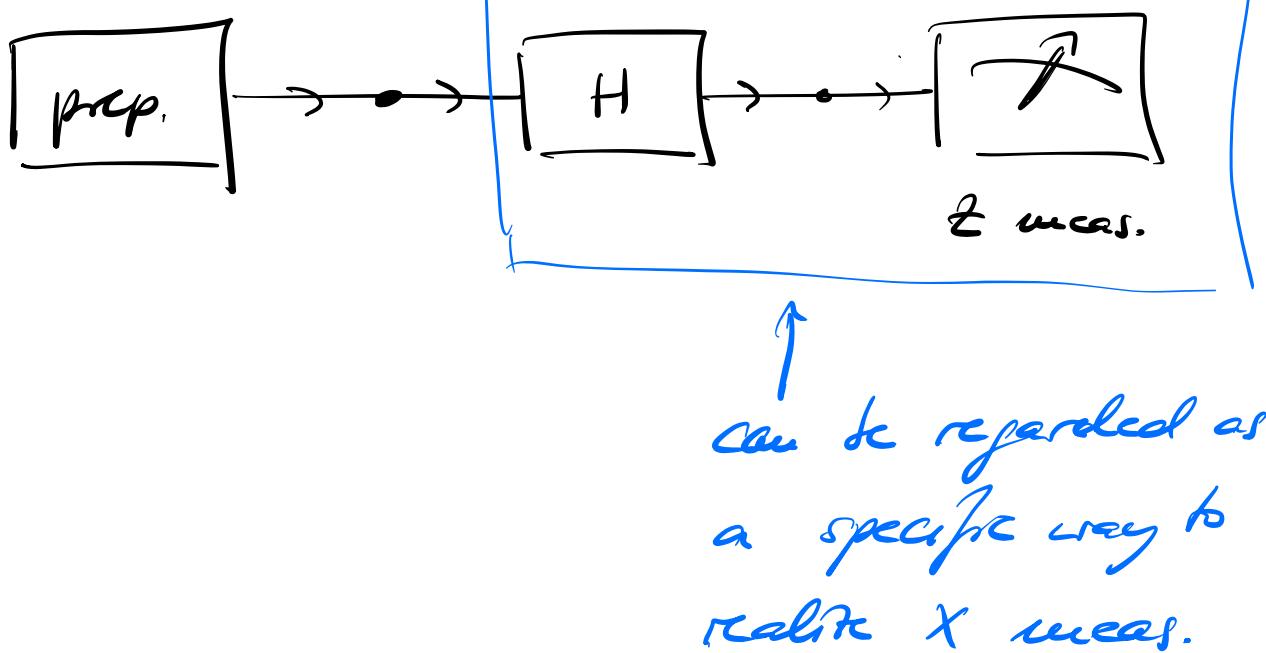
outcome 0 w/ $p_0 = \frac{|\alpha + \beta|^2}{2}$

outcome 1 w/ $p_1 = \frac{|\alpha - \beta|^2}{2}$

\Rightarrow corresponds to meas. outcome in X-basis!



equals...



In fact, H transforms between X and Z eigenstates back and forth:

$$H = |+\rangle\langle 0| + |-\rangle\langle 1| = |0\rangle\langle +| + |1\rangle\langle -| = H^+$$

i.e.: $HXH = Z, \quad HZH = X \quad (\text{note } H^2 = I).$

Measurement on a bipartite state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Alice and Bob measure Z:

project onto $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

$$\Rightarrow P_{01} = P_{10} = \frac{1}{2}, \quad P_{00} = P_{11} = 0$$

Alice and Bob measure X :

project onto $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$:

(use $\langle +|0\rangle = \langle +|1\rangle = \langle -|0\rangle = \frac{1}{\sqrt{2}}, \langle -|1\rangle = -\frac{1}{\sqrt{2}}$)

$$|\langle ++| \Psi \rangle|^2 = \left| \frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \right|^2 = 0$$

$$|\langle +-| \Psi \rangle|^2 = \left| -\frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \right|^2 = \frac{1}{2}$$

$$|\langle -+| \Psi \rangle|^2 = \dots = \frac{1}{2}$$

$$|\langle --| \Psi \rangle|^2 = \dots = 0$$

\Rightarrow perfect anti-correlation!

In fact, outcomes anti-correlated for same measurement in any basis! (\rightarrow homework)

(But the outcomes of A or B alone are completely random.)

But: Alice measures X , Bob Z :

$$|\langle +0 | \psi \rangle|^2 = \left| -\frac{1}{2} \right|^2 = \frac{1}{4}$$

$$|\langle +1 | \psi \rangle|^2 = \left| +\frac{1}{2} \right|^2 = \frac{1}{4}$$

$$|\langle -0 | \psi \rangle|^2 = \dots = \frac{1}{4}$$

$$|\langle -1 | \psi \rangle|^2 = \dots = \frac{1}{4}$$

Outcomes of A & B are completely independent.

d) The Bloch sphere:

Consider state of one qubit:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

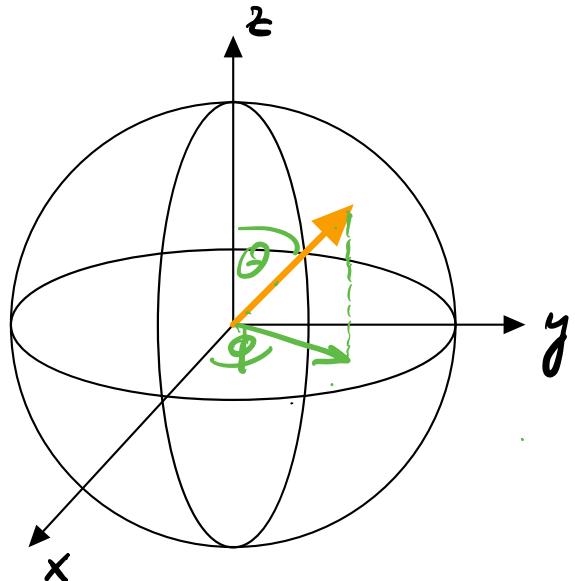
$$|\alpha|^2 + |\beta|^2 = 1$$

Define $\theta \in [0; \pi]$: $\cos \frac{\theta}{2} = |\alpha|$; $\sin \frac{\theta}{2} = |\beta|$.

Let $\alpha = e^{ix} |\alpha|$; $\beta = e^{i(x+\phi)} |\beta|$.

Then $|\psi\rangle = \underbrace{e^{i\chi}}_{\substack{\uparrow \\ \text{irrelevant}}} \left(\underbrace{\cos \theta/2 |0\rangle + e^{i\phi} \sin \theta/2 |1\rangle}_{\substack{\downarrow \\ \text{global phase}}} \right)$

depict on sphere:



"Bloch sphere"

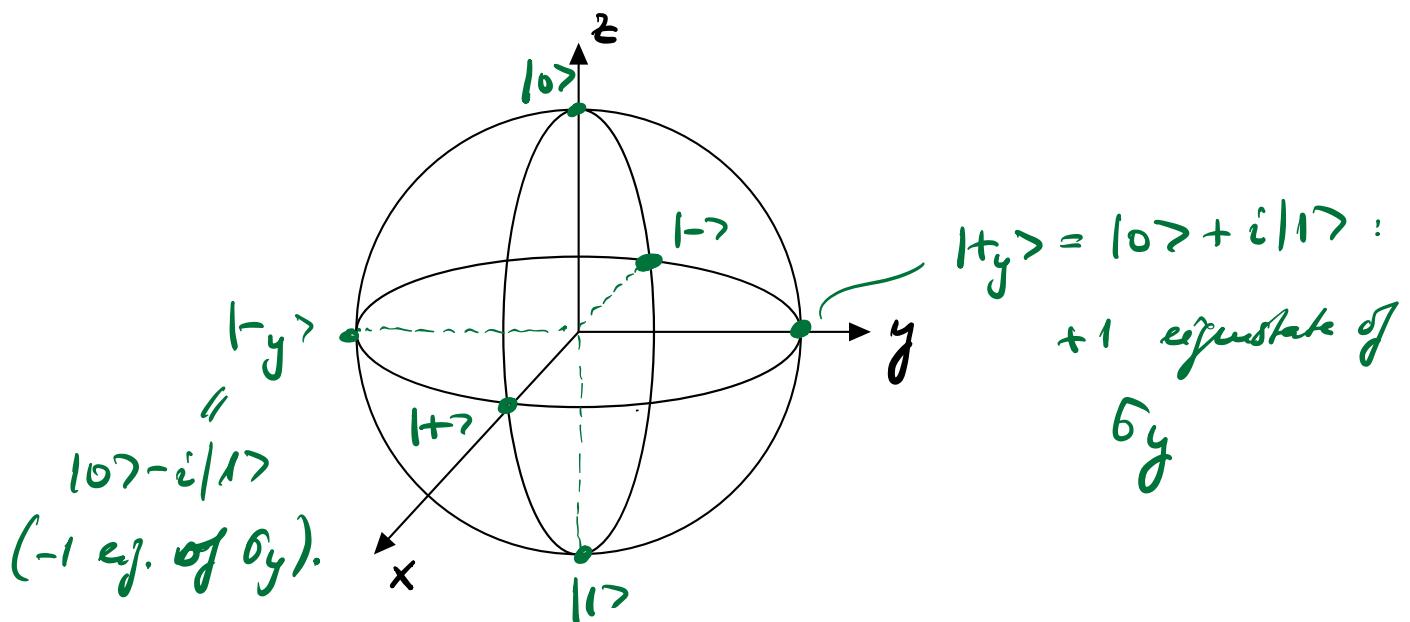
unit vector \vec{r} , $|\vec{r}| = 1$,
with angle θ to z axis
& angle ϕ in eq. plane
for x axis.

1-to-1 correspondence b/w states of qubit and points \vec{r} ("Bloch vector") on the sphere ("Bloch sphere").

Powerful visualization for qubit states.

Properties (just stated \rightarrow proof of Heisenberg):

important states:



General hermitian matrix w/ eigenvalues $\pm l$ is of

the form $M = \underbrace{\vec{m} \cdot \vec{\sigma}}$, $\vec{m} \in \mathbb{R}^3$, $|\vec{m}| = 1$.

Denote $m_1\sigma_1 + m_2\sigma_2 + m_3\sigma_3$

$$\equiv m_x\sigma_x + m_y\sigma_y + m_z\sigma_z.$$

(Lesson: $\{I, \sigma_x, \sigma_y, \sigma_z\}$ is a basis of herm. matrices over \mathbb{R} , and all matx. over $\mathbb{C} \rightarrow$ cf. Heisenberg.)

\Rightarrow eigenstates $\pm l$ of M have block vectors $\pm \vec{m}$.

Orthogonal states are anti-parallel or ~~stark~~ ^{Chapter II, pg 35} parallel.

For a state $|4\rangle$ w/ Bloch vector \vec{r} ,

$$\langle 4 | \sigma_i | 4 \rangle = r_i,$$

i.e. $|4\rangle$ can be interpreted as a $\text{spin } -\frac{1}{2}$ particle
in direction \vec{r} (note that $\vec{\Sigma} = \frac{1}{2} \vec{\sigma}$ is the
spin operator).

Measurement of qubit:

Observable w/ eigenvalues $\pm \epsilon$ (most gen. up to shift
& rescaling.) is of form $\Pi = \vec{m} \cdot \vec{\sigma}$, with
eigen space projectors $E_{\pm \epsilon} = \frac{I \pm \vec{m} \cdot \vec{\sigma}}{2}$.

Prob. for outcome $\pm \epsilon$ is then

$$P_{\pm \epsilon} = \langle 4 | E_{\pm \epsilon} | 4 \rangle = \frac{1 \pm \vec{m} \cdot \vec{r}}{2}.$$

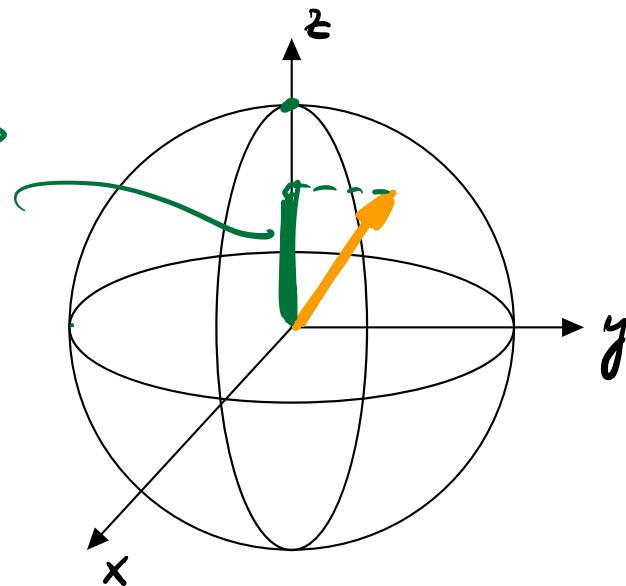
(Note: $\vec{m} \cdot \vec{r}$ is project of \vec{r} onto axis \vec{m} !)

E.g. meas in 2 basis:

$$\text{Result } \pm 1 \text{ w prob. } P_{\pm} = \frac{1 \pm r_z}{2}$$

Projection onto \vec{m} : $\vec{m} \cdot \vec{r}$

Probability changes
linearly along z axis
for 1 to 0, or 0 to 1.



$$\Leftrightarrow P = \frac{1 + \vec{m} \cdot \vec{r}}{2},$$

Evolution:

Unitaries on qubits are of the form

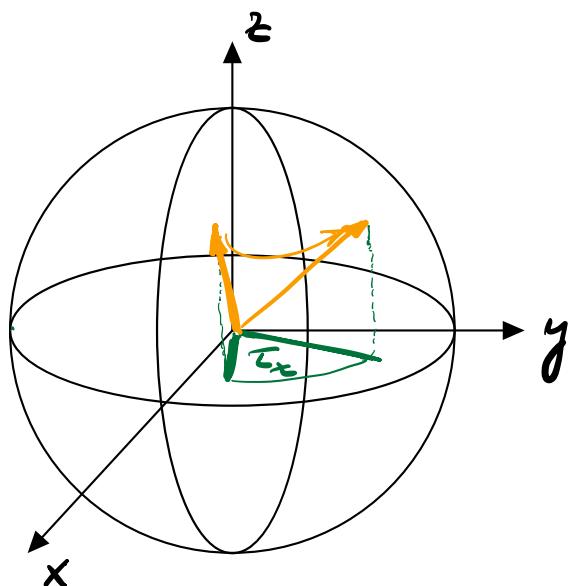
$$U = e^{i\chi} \exp \left[-i \vec{\tau} \cdot \vec{\sigma}/2 \right]$$

(Proof idea: Go from U to generator $G = G^+$, $U = e^{-iG}$,
and write G as $G = \vec{m} \cdot \vec{\sigma} + c.I.$)

On Bloch sphere:

U rotates Bloch vector by angle $|\vec{\tau}|$ about the axis $\vec{\tau}/|\vec{\tau}|$.

E.g.: $U_z(\tau) = \exp(-i\tau_z \sigma_z/2)$:



This is a consequence of the double cover

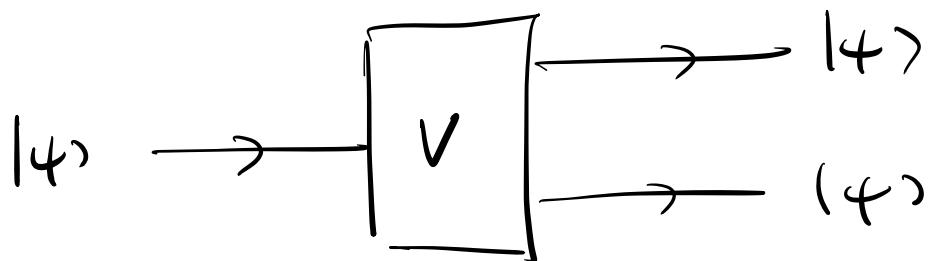
$$\mathfrak{su}(2)/\mathbb{Z}_2 \cong \mathfrak{so}(3)$$

(The \cdot/\mathbb{Z}_2 comes from the fact that a 2π rotation gives $\exp(-2\pi i \sigma_z/2) = -I$)
 (or other $\vec{\tau} \cdot \vec{\sigma}, |\vec{\tau}|=1$.)

Question: What rotation is $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$?

e) A fundamental consequence: The no-cloning theorem

Given an unknown quantum state $|q\rangle \in \mathcal{H}$, can we build a device which does



i.e. a transformation

$$V: \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$$

$$|q\rangle \mapsto |q\rangle \otimes |q\rangle \quad ?$$

How to build V - dim. of \mathcal{H} and $\mathcal{H} \otimes \mathcal{H}$ are different!

→ Add an auxiliary system ("ancilla") of same dimension:

$$U: \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$$

$$|q\rangle \otimes |0\rangle \mapsto |q\rangle \otimes |q\rangle$$

any suitable fiducial state

Note: $V := U(I_A \otimes |0\rangle_B)$ is an isometry:

$$V^+ V = (|0\rangle_B \otimes I_A) \underbrace{U^+ U}_{I_{AB}} (I_A \otimes |0\rangle_B) \\ = |0\rangle_B I_{AB} |0\rangle_B = I_A$$

No-cloning Theorem:

Quantum information cannot be copied, i.e. a

$$U : |+\rangle \otimes |0\rangle \mapsto |+\rangle \otimes |+\rangle \quad \otimes$$

cannot exist.

$$\text{Proof: } U(|0\rangle \otimes |0\rangle) \stackrel{\otimes}{=} |0\rangle \otimes |0\rangle$$

$$U(|1\rangle \otimes |0\rangle) \stackrel{\otimes}{=} |1\rangle \otimes |1\rangle$$

$$\Rightarrow U\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle\right) = \frac{|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle}{\sqrt{2}} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

But, from (*):

$$U\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle\right) = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$= \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{\sqrt{2}}$$

→ Contradiction!

→ U cannot exist (note: we only used linearity!) 18

Quantum information cannot be copied!

But: A classical copier is consistent w/
quantum theory, i.e. a device

$$U: |ii\rangle \otimes |0\rangle \mapsto |ii\rangle \otimes |ii\rangle$$

for the comp. basis, or any other ONS $|i\rangle$.

(Proof: Homework)