

3. The Schmidt decomposition & purifications

a) The Schmidt decomposition

Consider a bipartite state $|4\rangle_{AB}$, and let

$$\text{tr}_B |4\rangle\langle 4| = \rho_A = \sum_i p_i |q_i\rangle\langle q_i| \text{ be the}$$

eigenvalue decomposition (including $p_i=0$),
i.e. $\{|q_i\rangle_A\}$ is an ONS.

Let $|x_i\rangle_B$ be any ONS of B , and expand

$$|4\rangle_{AB} = |q_i\rangle_A \otimes |x_j\rangle_B:$$

$$|4\rangle_{AB} = \sum c_{ij} |q_i\rangle_A |x_j\rangle_B$$

$$= \sum |q_i\rangle_A |\tilde{x}_i\rangle_B$$

$$\text{with } |\tilde{x}_i\rangle_B := \sum_j c_{ij} |x_j\rangle_B.$$

↑ no ONS etc. (a priori)

We have

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$$\sum p_i |a_i X_{a_i}\rangle = \text{tr}_0 |4 X_4\rangle$$

$$= \text{tr}_S \left[\sum_{ii'} |a_i X_{a_i}\rangle_A \otimes |\tilde{b}_{i'} X_{\tilde{b}_{i'}}\rangle_B \right]$$

$$= \sum_{ii'} |a_i X_{a_i}\rangle \text{tr} \left[|\tilde{b}_{i'} X_{\tilde{b}_{i'}}\rangle_B \right]$$

$$= \sum \langle \tilde{b}_{i'} | \tilde{b}_i \rangle |a_i X_{a_i}\rangle$$

Since the $|a_i X_{a_i}\rangle$ are lin. indep.

(in fact, H-S-orthonormal!), we must have

$$\langle \tilde{b}_{i'} | \tilde{b}_i \rangle = p_i \delta_{ii'} !$$

\Rightarrow For all i s.t. $p_i \neq 0$:

$$|\tilde{b}_i\rangle_B := \frac{1}{\sqrt{p_i}} |\tilde{b}_i\rangle_B \text{ are } \underline{\text{orthonormal!}}$$

\Rightarrow

$$|\Psi\rangle_{AB} = \sum_{i:p_i \neq 0} \sqrt{p_i} |a_i\rangle_A |\tilde{b}_i\rangle_B$$

with $\{|a_i\rangle_A\}$, $\{|\tilde{b}_i\rangle_B\}$ orthonormal!

(Note: Can be padded w/ $i: p_i=0$ as long as $\lambda_i \neq 0$ chapter II, pg 63
If δ is large enough.)

Definition: A decomposition $|4\rangle_{AB} = \sum p_i |a_i\rangle_A |b_i\rangle_B$ into ONS $\{|a_i\rangle_A\}$, $\{|b_i\rangle_B\}$, $p_i \geq 0$, is called Schmidt decomposition of $|4\rangle_{AB}$, with Schmidt coefficients p_i . The number of non-zero p_i 's is called Schmidt rank (or Schmidt number).

Theorem: Any bipartite state $|4\rangle_{AB}$ has a Schmidt decomposition, i.e. there exist p_i and ONS $\{|a_i\rangle_A\}$ and $\{|b_i\rangle_B\}$ s.t. $|4\rangle_{AB} = \sum p_i |a_i\rangle_A |b_i\rangle_B$.

(Proof: Construction from before.)

Note: $p_B = \text{tr}_A |4\rangle \langle 4| = \sum_i p_i |\lambda_i|$, and (by construction) $p_A = \sum_i p_i |a_i\rangle \langle a_i|$.

\Rightarrow (i) $\{|a_i\rangle_A\}$ and $\{|b_i\rangle_B\}$ are the eigenbases of p_A & p_B (except $p_i=0$), respectively.

(ii) p_A & p_B have the same eigenvalues (!).

(iii) If the p_i are non-degenerate, the

Schmidt decoupl. can be found by
partly up the eigenvectors of $P_A \otimes P_B$!
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(iv) $\{|a_i\rangle_A\}$ and $\{|s_i\rangle_B\}$ ($p_i \neq 0$) are ONBs
of $\text{supp } P_A$ and $\text{supp } P_B$, respectively.

(Note: If $\dim A = \dim B$, we can include
the zero eigenvalues & extend $\{|a_i\rangle_A\}$,
 $\{|s_i\rangle_B\}$ to ONBs of the full space.)

Notational conventions:

Often, the bases $\{|a_i\rangle_A\}$ and $\{|s_i\rangle_B\}$
are simply denoted as

$$|i\rangle_A := |a_i\rangle_A, \text{ and}$$

$$|i\rangle_B := |s_i\rangle_B.$$

These are not the computational basis, and
generally different bases on $A \otimes B \Rightarrow \underline{\text{CAREFUL!!}}$

How is the Schmidt decomposition related to
expansion of $|q\rangle$ in a different pair of basis
 $\{|x_i\rangle_A\}$, $\{|y_j\rangle_B\}$?

$$\begin{aligned} \text{We have } |q\rangle &= \sum c_{ij} |x_i\rangle_A |y_j\rangle_B \\ &= \sum p_k |\alpha_k\rangle_A |\beta_k\rangle_B. \end{aligned} \quad (*)$$

\Rightarrow There exist matrices $U = (u_{ik})$, $V = (v_{jk})$ s.t.

$$|\alpha_k\rangle_A = \sum u_{ik} |x_i\rangle_A, \quad |\beta_k\rangle_B = \sum \overline{v_{jk}} |y_j\rangle_B,$$

$$\begin{aligned} \text{and } \delta_{kl} &= \langle \alpha_k | \alpha_l \rangle = \sum_{ij} \overline{u_{ik}} u_{il} \underbrace{\langle x_i | x_j \rangle}_{= \delta_{ij}} \\ &= \sum \overline{u_{ik}} u_{il} = (U^\dagger U)_{kl}, \end{aligned}$$

$$\text{and equally } (V^\dagger V)_{kl} = \delta_{kl}$$

$\Rightarrow U, V$ are isometries.

If we neglect this in (*):

$$\sum_{ij} c_{ij} |x_i\rangle_A \langle y_j|_B = \sum_{ij} \sum_k \sqrt{p_k} u_{ik} \bar{v}_{jk} |x_i\rangle_A \langle y_j|_B$$

↑
lin. indep!

$$\Rightarrow c_{ij} = \sum_k \underbrace{\sqrt{p_k} u_{ik}}_{\parallel \ddot{i}_k} \bar{v}_{jk} \quad \text{this or}$$

Theorem:

Any matrix $C = (c_{ij})$, $i=1, \dots, n$, $j=1, \dots, m$, can be written in the form

$$C = U \cdot D \cdot V^+, \text{ or}$$

$$c_{ij} = \sum_{k=1}^r \lambda_k u_{ik} \bar{v}_{jk}$$

with $r = \text{rank}(C) \leq n, m$, $\lambda_k > 0$, and

$$U = (u_{ik}), \quad V = (v_{jk}), \quad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & \ddots & \lambda_r \end{pmatrix},$$

$i=1, \dots, n$; $k=1, \dots, r$; $j=1, \dots, m$, and

$$U^+ U = I_r, \quad V^+ V = I_r \quad \text{so we have.}$$

This is called the singular value decomposition (SVD) of C ,
with singular values λ_i :

If the λ_i are ordered descendingly,
 $\lambda_1 \geq \lambda_2 \geq \dots$, U and V are unique up
to joint rotations in subspaces with different
singular values.

Alternatively, one can choose U and V
square $m \times n$ and $n \times n$ unitary, and

$$D = \begin{pmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_r & \\ \hline & & & 0 \end{pmatrix},$$

but the additional degrees of freedom are
arbitrary.

Proposition: Any two states $|\phi\rangle, |\psi\rangle$ with identical Schmidt coefficients $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0$ are related by local unitaries, i.e.

$$\exists U, V: |\phi\rangle = (U \otimes V) |\psi\rangle.$$

Thus: The ordered Schmidt coefficients $\lambda_1 \geq \lambda_2 \geq \dots$ encode all non-local properties.

Proof: $|\phi\rangle = \sum_{i=1}^r \lambda_i |\phi_i^A\rangle \otimes |\phi_i^B\rangle$

$$|\psi\rangle = \sum_{i=1}^r \lambda_i |\psi_i^A\rangle \otimes |\psi_i^B\rangle$$

$\{|\phi_i^A\rangle\}, \{|\psi_i^A\rangle\}$ orthonormal $\Rightarrow \exists U: |\phi_i^A\rangle = U |\psi_i^A\rangle$ th.

$\{|\phi_i^B\rangle\}, \{|\psi_i^B\rangle\}$ orthonormal $\Rightarrow \exists V: |\phi_i^B\rangle = V |\psi_i^B\rangle$ th.

□

(Note that also vice versa, the Schmidt coefficients cannot be changed at all by local unitaries - all information in them is non-local.)

6) Purifications (part)

Reminder: Given ρ_A on A , a state $| \psi \rangle \in \mathcal{H}_A \otimes \mathcal{H}_S$

s.t. $\text{tr}_S |\psi\rangle\langle\psi| = \rho_A$ is called a purification of ρ_A .

Given two purifications

$$\begin{aligned} |\phi\rangle &\in \mathcal{H}_A \otimes \mathcal{H}_B \\ |\psi\rangle &\in \mathcal{H}_A \otimes \mathcal{H}_{B'} \end{aligned}$$

↑
potentially
different spaces

of ρ_A , what is their relation?

Write both $|\phi\rangle$ and $|\psi\rangle$ in their Schmidt form
(not a basis baf.):

$$|\phi\rangle = \sum \lambda_i |\phi_i^A\rangle |\phi_i^B\rangle$$

$$|\psi\rangle = \sum \mu_i |\psi_i^A\rangle |\psi_i^{B'}\rangle$$

(wlog λ_i, μ_i descending; then $B' \geq \dim B$.)

We have

$$\sum \lambda_i^2 |\phi_i^A X \phi_i^A| = \text{tr}_B |\phi X \phi| = p_A = \\ = \text{tr}_B |\psi X \psi| = \sum \mu_i^2 |\psi_i^A X \psi_i^A|$$

$|\phi_i^A\rangle, |\psi_i^A\rangle$ orthonormal

\Rightarrow If λ_i, μ_i non degenerate, then

$$\lambda_i = \mu_i, \quad |\phi_i^A\rangle = |\psi_i^A\rangle \quad \forall i$$

(If degen.: Schmidt decoupl. can be constructed from any eigen decomposition $\sum \lambda_i |\phi_i X \phi_i|$ of p_A - see Sec. a) - so we can construct A with the same eigenvectors $|\phi_i^A\rangle = |\psi_i^A\rangle$.)

Now construct a $U: \mathcal{H}_B \rightarrow \mathcal{H}_B'$

s.t. $|\phi_i^B\rangle \mapsto |\psi_i^{B'}\rangle$.

U is a unitary between $\text{span} \{|\phi_i^B\rangle\}$

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and space $\{|\psi_i^{B'}\rangle\}$. If $\dim B' > \dim B$,
it can be extended to an isometry. Then:

$$|\psi\rangle = (\mathbb{I}_A \otimes U_B) |\phi\rangle.$$

Theorem: All purifications of a given P_A are related by a unitary (or isometry) on the purifying system.

Note: This is closely linked to the unitary / isometric ambiguity of the ensemble decomposition:

Any ensemble $P = \sum p_i |\phi_i X_{\phi_i}\rangle$ is related

to a purification $|\psi\rangle = \sum \sqrt{p_i} |\phi_i\rangle_A |i\rangle_B$

from where it can be obtained by measuring B in the computational basis. (\rightarrow Homework #7)