4. POVM measurements

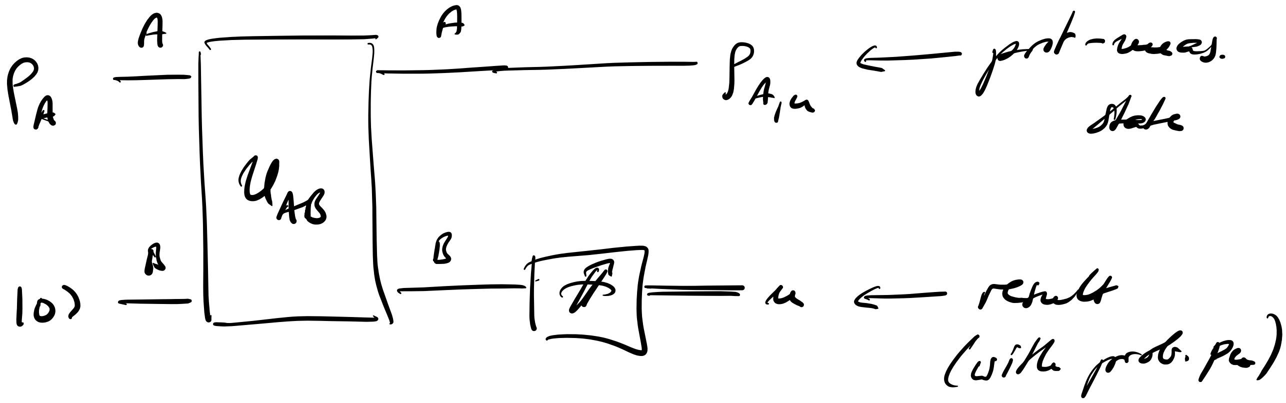
Seen previously: Adding a 2nd system B gives more rich situation.

Thus natural question: What measurements can we do by adding an extra system?

Idea: i) Add auxiliary system ("ancilla") B in state $|0\rangle$

ii) Act w/ unitary U_{AB} on system + ancilla

iii) measure B in $\{|0\rangle, |1\rangle, \ldots, |d_B-1\rangle\}$



Analyze scheme:

Post-meas. state (unnormalized) is:

$$\tilde{\rho}_n^A = \langle n|_B\, U \left(\rho_A \otimes |0\rangle\langle 0|_B\right) U^\dagger |n\rangle_B$$

$$= \langle u|_B \, U |0\rangle_B \;\; \rho_A \;\; \langle 0|_B \, U^\dagger |u\rangle_B$$

$$= \Pi_u \, \rho_A \, \Pi_u^\dagger ,$$

where we have defined

$$\Pi_u := \langle u|_B \, U |0\rangle_B \equiv (I_A \otimes \langle u|_B) \, U \, (I_A \otimes |0\rangle_B)$$

Then, $p_u = \mathrm{tr}\, \tilde{\rho}_u^A = \mathrm{tr}(\Pi_u \rho_A \Pi_u^\dagger) = \mathrm{tr}(\Pi_u^\dagger \Pi_u \rho_A)$,

is the probability for outcome u,

and $\rho_u^A = \frac{1}{p_u} \tilde{\rho}_u^A$ the post-measurement state.

It holds that

$$\sum \Pi_u^\dagger \Pi_u = \sum_u \langle 0|_B \, U^\dagger |u\rangle_B \langle u|_B \, U |0\rangle_B$$

$$= \langle 0|_B \, U^\dagger U |0\rangle_B$$

$$= I_A ,$$

and further $\Pi_u^\dagger \Pi_u \geq 0$.

(Note: The former implies

$$\sum p_u = \sum \mathrm{tr}(\Pi_u^\dagger \Pi_u \rho) = \mathrm{tr}\left(\sum \Pi_u^\dagger \Pi_u \rho\right) = \mathrm{tr}(\rho) = 1 \,).$$

Definition: A set $\{F_m\}$ of operators, $F_m \geq 0$, $\sum F_m = I$, is called a **positive operator-valued measure (POVM)**.

Note: $F_m := M_m^{+} M_m$ forms a POVM. If we only care about the post-meas. prob. $p_m = tr(F_m \rho)$, then the measurement is fully characterized by the POVM $\{F_m\}$.

Definition: A **POVM measurement** is given by a set of operators $\{M_m\}$ with $\sum M_m^{+} M_m = I$, with outcome probabilities $p_m = tr(M_m^{+} M_m \rho)$ and post-measurement states $\rho_m = \frac{1}{p_m} M_m \rho M_m^{+}$.

Alternative Definition: A POVM measurement is given by a set of operators $\{F_m\}$, $F_m \geq 0$, $\sum F_m = I$, with outcome probabilities $p_m = tr(F_m \rho)$.

Relation of the two definitions, & with the initial unitary + ancilla construction:

i) Can any $F_n \geq 0$ be written as $F_n = \Pi_n^\dagger \Pi_n$?

Yes – e.g., take $\Pi_n = \sqrt{F_n}$.

(Unique up to isometric degree of freedom, since

$\Pi_n = U_n \sqrt{\Pi_n^\dagger \Pi_n}$ (the polar decomposition).

ii) Can any POVM meas. $\{\Pi_n\}_{n=0}^{N-1}$, $\sum_{n=0}^{N-1} \Pi_n^\dagger \Pi_n = I$,

be realized via ancilla + unitary?

$$X := \begin{pmatrix} \Pi_0 \\ \Pi_1 \\ \vdots \\ \Pi_{N-1} \end{pmatrix}$$

$\sum \Pi_n^\dagger \Pi_n = I \iff X$ has orthogonal columns

$\implies X$ can be extended to a unitary U by adding further columns,

$$\mathcal{U} = \begin{array}{c} \\ \langle 0|_B \\ \langle 1|_B \\ \vdots \\ \langle d-1|_B \end{array} \begin{array}{c} \begin{array}{ccc} |0\rangle_B & |1\rangle_B & \cdots \end{array} \\ \begin{pmatrix} M_0 & \cdots & \cdots \\ M_1 & \cdots & \cdots \\ \vdots & & \\ M_{N-1} & \cdots & \cdots \end{pmatrix} \end{array}$$

... this can be understood as a unitary acting on system + ancilla B with dim. $d_B = N$.

$$\Longrightarrow \quad \langle n|_B \, \mathcal{U} \, |0\rangle_B = M_n.$$

$\Longrightarrow$ Any POVM meas. $\{M_n\}$ can be realized by adding ancilla, doing a unitary $\mathcal{U}$ on system + ancilla, and projectively measuring ancilla in $\{|0\rangle, \dots, |d_B - 1\rangle\}$ basis.

This is also known as <u>Naimark's Theorem</u>.

<u>Note:</u> The "old-style" measurements where the $M_n = E_n$ (or equivalently $F_n \equiv E_n$) are also called <u>projective measurements</u>.

Is this the most general type of measurement?

i) Minimal requirements for q.m. measurements:

Measurements are linear functionals

$$\rho \longmapsto p_n(\rho) \, ,$$

which map states to outcome probabilities,

such that

$$p_n(\rho) \geq 0 \qquad \forall \rho \geq 0, \, tr(\rho) = 1$$

and

$$\sum p_n(\rho) = 1 \qquad \forall \rho \geq 0, \, tr(\rho) = 1.$$

ii) Linear functionals $\rho \mapsto f(\rho)$ $(\rho \geq 0)$ can be uniquely extended (over $\mathbb{C}$) to all matrices X,

$$X \longmapsto f(X), \text{ as } X = \underbrace{\frac{X + X^\dagger}{2}} + i \cdot \underbrace{\frac{X - X^\dagger}{2i}} ,$$

hermitian

and any hermitian $H = \underbrace{P}_{\geq 0} - \underbrace{N}_{\geq 0}$

(e.g. P, N from pos./neg. eigenvalues).

iii) linear functionals $X \longmapsto p_u(X)$ are of the form $p_u(\rho) = tr(F_u \rho)$,

$$\left(\text{E.g. from } p_u(\rho) = p_u\left(\sum \rho_{ij} |i\rangle\langle j|\right) = tr\left(\underbrace{\left[\sum p_u(|i\rangle\langle j|)\, |j\rangle\langle i|\right]}_{=: F_u} \rho \right) \right)$$

and F_u is unique (as $tr(A^\dagger B)$ is a scalar product.)

iv) $tr(I\rho) = tr(\rho) = 1 = \sum p_u(\rho) = tr\left(\sum F_u \rho\right)$

$$\Rightarrow \sum F_u = I.$$

v) $F_u \geq 0$ — otherwise, $\exists |\phi\rangle : \langle\phi|F_u|\phi\rangle \ngeq 0$, and thus for $\rho = |\phi\rangle\langle\phi|$,

$$p_u(\rho) = tr\left[F_u \cdot |\phi\rangle\langle\phi|\right] = \langle\phi|F_u|\phi\rangle \ngeq 0$$

<u>Thus:</u> POVM measurement is the <u>most general linear measurement</u> on density matrices.