III. Entanglement

1. Introduction

Consider a bipartite state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.

We say that $|\psi\rangle$ is a product state if it can be written as

$$|\psi\rangle = |\phi^A\rangle \otimes |\phi^B\rangle$$

for some $|\phi^A\rangle \in \mathcal{H}_A$, $|\phi^B\rangle \in \mathcal{H}_B$.

If $|\psi\rangle$ is not a product state, we say that $|\psi\rangle$ is entangled.

How can we tell if a state is a product state or entangled?

Consider Schmidt decomposition of $|\psi\rangle$:

$$|\psi\rangle = \sum_{i=1}^{r} \lambda_i \, |\phi_i^A\rangle \otimes |\phi_i^B\rangle$$

orthonormal

with r the Schmidt rank.

- $r = 1 \Rightarrow |\psi\rangle = |\phi_1^A\rangle \otimes |\phi_1^B\rangle \Rightarrow |\psi\rangle$ product.
- $|\psi\rangle$ product $\Rightarrow |\psi\rangle = |\phi^A\rangle \otimes |\phi^B\rangle$ is Schmidt decomposition $\Rightarrow r = 1$.

__I.e.:__ __Product states__ have __Schmidt rank $r=1$.__

__Entangled states__ have __Schmidt rank $r \geq 2$.__

In particular, states such as

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle - |1\rangle|0\rangle\right)$$

$$\text{or} \quad |\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i,i\rangle$$

are entangled.

__Product states__: Can describe any action of A & B __independently__, and state

$$|\psi\rangle = |\phi^A\rangle \otimes |\phi^B\rangle \longmapsto (M_A \otimes N_B)|\psi\rangle$$
$$= M_A|\phi^A\rangle \otimes N_B|\phi^B\rangle$$

stays a __product state__.

Entangled states:

Generally cannot describe actions of A & B independently – e.g., meas. outcomes in Schmidt basis will be perfectly correlated.

Can we use these correlations for non-trivial tasks?

Questions in entanglement theory:

- How non-classical are entangled states?
- What can we do with entangled states? (i.e., are they a resource?)
- How can we quantify the amount of entanglement (e.g. in terms of usefulness)?
- Are there different types of entangled states?
- How can we manipulate entangled states?
- What about mixed state entanglement?