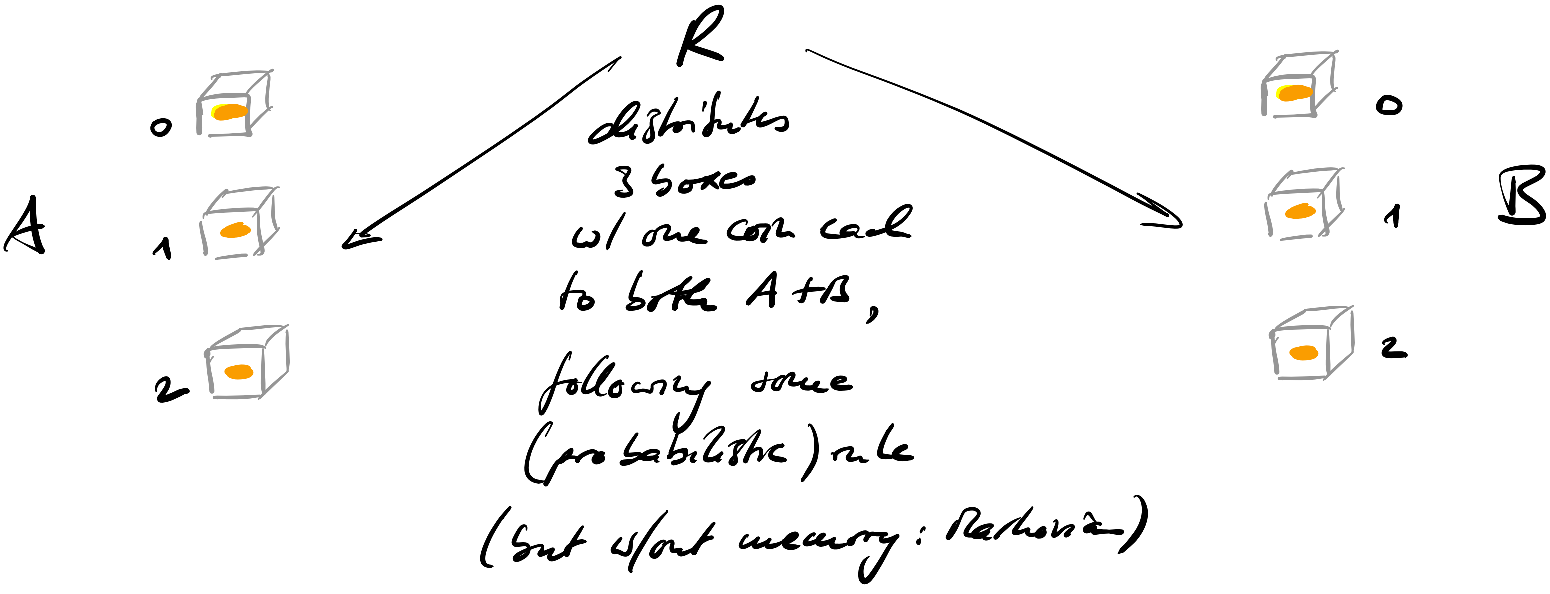
2. Bell inequalities

"How non-classical are entangled states?"

a) The Bell inequality

Consider the following game played by Alice + Bob with coins, prepared by a referee R:



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- A & B play many rounds w/ the referee. In each round, A & B each get 3 coins in closed boxes (labelled 0,1,2), prepared according to some rule (deterministic or random, but the same indep. rule in every rd.) by R.

- In each rd., A & B can look at only <u>one coin each</u> ($x=0,1,2$; $y=0,1,2$). We denote heads $=+1$

and tails $= -1$; and the obtained results by

$a_x = \pm 1$; $b_y = \pm 1$.

- After that, the boxes are collected by the referee, and a new round starts.

(ii) By repeatedly measuring the same box, $x=y$, A & B observe: They always get the <u>same outcome</u>, i.e.

$$\boxed{a_x = b_x}$$

(iii) A & B are smart: They can use this to cheat the referee and obtain the value of <u>two coins</u> in <u>a single round</u>.

<u>Idea:</u> A checks coin x, B checks coin $y = x' \neq x$.

Then, since $a_{x'} = b_{x'}$, they know <u>a_x and $a_{x'}$</u> in the same round!

This <u>clearly works</u> in a classical scenario (i.e., with coins – we will formalize this later).

Consequence: A & B can use this to estimate

$$p(a_x = b_{x'}) = p(a_x = a_{x'})$$

using that $a_{x'} = b_{x'}$
(requires classical corrs)

Observation:

$$p(a_0 = a_1) + p(a_1 = a_2) + p(a_2 = a_0) \geq 1,$$

$$\frac{N(a_0 = a_1)}{N_{tot}} + \frac{N(a_1 = a_2)}{N_{tot}} + \frac{N(a_2 = a_0)}{N_{tot}}$$

# of rounds where $a_0 = a_1$

total # of rounds

since in each round, at least two corrs must be equal (or all 3)!

Using that (classically) $a_x = b_x$:

$$\boxed{p(a_0 = b_1) + p(a_1 = b_2) + p(a_2 = b_0) \geq 1} \quad (*)$$

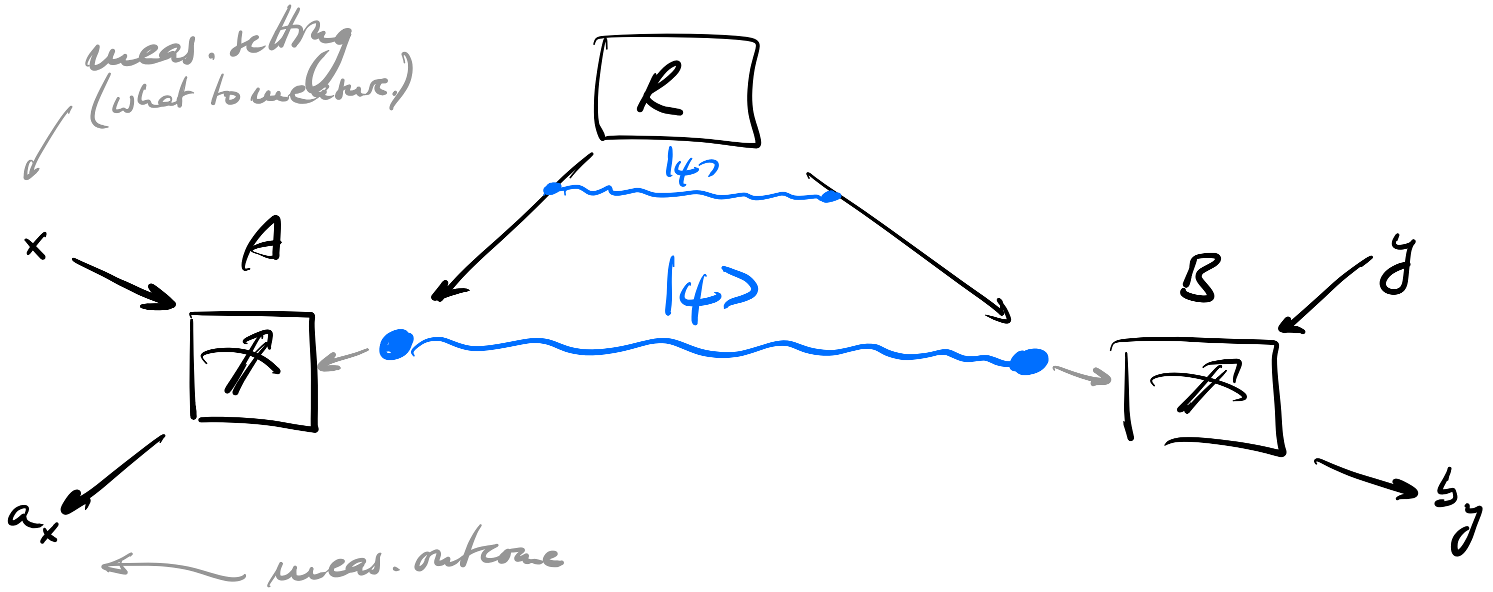
is satisfied for any classical theory (which satisfies $a_x = b_x \; \forall x$).

(*) is called **Bell inequality**!

(Note: Bell inequalities are inequalities satisfied by classical theories, and have a priori nothing to do with quantum theory!)

But: In a suitable quantum mechanical version of the game, ⊛ is violated!

Quantum version of the game:



- R distributes bipartite state $|\psi\rangle$.
- A & B perform a measurement dep. on x/y, w/ outcome a_x/b_y.

(I.e.: Which box to open = which meas. to perform — A & B can always only do <u>one</u> meas. each on original state).

<u>Setup:</u> • $|\psi\rangle = |\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

• A & B do projective measurement along axes $\vec{u}_x$ and $\vec{m}_y$, i.e. they measure the operators $\vec{u}_x \cdot \vec{\sigma}^A$ and $\vec{m}_y \cdot \vec{\sigma}^B$.

• We will specify $\vec{u}_x$ and $\vec{m}_y$ later on.

It holds that $(\underbrace{\vec{\sigma}^A}_{\equiv \vec{\sigma}_A \otimes I_B} + \underbrace{\vec{\sigma}^B}_{\equiv I_A \otimes \vec{\sigma}^B})|\psi^-\rangle = 0$

(i.e. $(\sigma^A_\alpha + \sigma^B_\alpha)|\psi^-\rangle = 0 \quad \forall \alpha = x, y, z$; by direct inspection, or since $|\psi^-\rangle$ has spin 0).

$\Rightarrow \langle\psi^-|(\vec{\sigma}^A \cdot \vec{u})\underbrace{(\vec{\sigma}^B \cdot \vec{m})|\psi^-\rangle}_{= -\vec{\sigma}^A \cdot \vec{m}|\psi\rangle \text{ from the above}} =$

$= -\langle\psi^-|(\vec{\sigma}^A \cdot \vec{u})(\vec{\sigma}^A \cdot \vec{m})|\psi^-\rangle$

$$= - \sum_{kl} u_k m_l \underbrace{\langle \psi^- | \sigma_k^A \sigma_l^A | \psi^- \rangle}_{= \mathrm{tr}\left[\rho_A \sigma_k^A \sigma_l^A\right]}$$

$$= \mathrm{tr}\left[\rho_A \sigma_k^A \sigma_l^A\right]$$

$\rho_A = \frac{1}{2} I$, e.g. from Schmidt dec.

$$= \frac{1}{2} \mathrm{tr}\left[\sigma_k^A \sigma_l^A\right] = \delta_{kl}$$

$$= - \sum_k u_k m_k = - \vec{u} \cdot \vec{m} = - \cos\theta$$

angle between $\vec{u}$ and $\vec{m}$.

Measurement along $\vec{u}$: Meas. operators (projectors)

$$E_{\pm 1}(\vec{u}) = \frac{1}{2} \left(I \pm \vec{u} \cdot \vec{\sigma} \right)$$

Let $p(a,b)$, $a,b = \pm 1$ denote prob. to get outcomes a & b, respectively, for A & B. Then,

$$p(\pm 1, \pm 1) = \langle \psi^- | E_\pm^A(\vec{u}) \, E_\pm^B(\vec{m}) | \psi^- \rangle$$

$$= \frac{1}{4} \langle \psi^- | \underbrace{I}_{\equiv 1} \pm \underbrace{\vec{u} \cdot \vec{\sigma}^A}_{\equiv 0 \text{ since } \rho_A = \frac{1}{2} I} \pm \underbrace{\vec{m} \cdot \vec{\sigma}^B}_{\equiv 0} + \underbrace{(\vec{u} \cdot \vec{\sigma}^A)(\vec{m} \cdot \vec{\sigma}^B)}_{\equiv -\cos\theta} | \psi^- \rangle$$

$$= \frac{1}{4} \left(1 - \cos\theta \right).$$

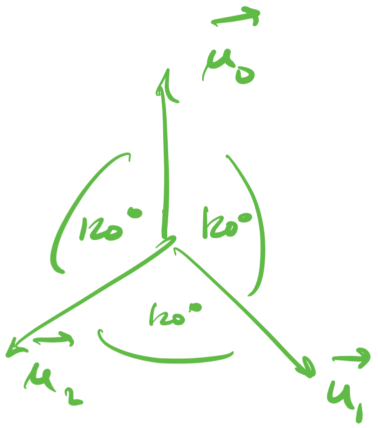
and

$$p(\pm 1, \mp 1) = \frac{1}{4}(1+\cos\theta).$$

$$\Rightarrow \text{prob}(a=b) = \frac{1}{2}(1-\cos\theta)$$

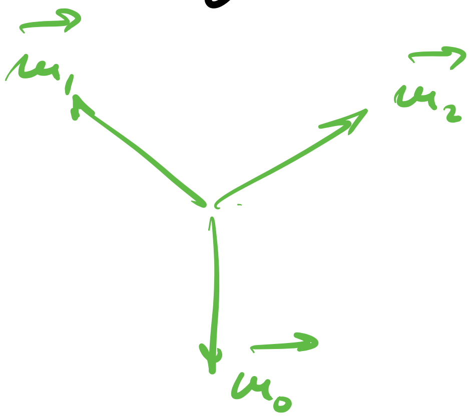
$$\text{prob}(a\neq b) = \frac{1}{2}(1+\cos\theta)$$

Now let A choose measurements $\vec{u}_0$, $\vec{u}_1$, $\vec{u}_2$:



in the XZ-plane,

and B along $\vec{m}_x = -\vec{u}_x$:



Then:

- $x = y$: $p(a_x = b_y) = \frac{1}{2}(1 - \cos 180°) = 1$

$\Rightarrow$ A & B always get <u>same result</u> when they measure "the same coin"

- $x \neq y$: $p(a_x = b_y) = \frac{1}{2}\left(1 - \underbrace{\cos(\pm 60°)}_{= 1/2}\right) = \frac{1}{4}$

$\Rightarrow$ $p(a_0 = b_1) + p(a_1 = b_2) + p(a_2 = b_0) = \frac{3}{4} < 1$

(while Bell ineq. stated ≥ 1 for class. theories!)

<u>Bell inequality violated!</u>

Formally, what were the <u>assumptions</u> of our <u>classical theory</u>?

(1) <u>Realism</u>: Outcomes of measurements are "elements of reality" – i.e., they have pre-determined values even prior to measurement.

(2) Locality: A & B's boxes cannot communicate once distributed.

⇒ Quantum mechanical predictions are incompatible with any local and realistic theory — we need to give up either locality or realism.

Bell inequalities can be used to certify that a system behaves quantum mechanically (i.e. non-classically): If we measure a violation of the Bell inequality — note that $p(a_x = b_y)$ can be estimated reliably by repeated measurements — we know that the system cannot be described classically and must thus be quantum mechanical.

b) The CHSH inequality

Bell's inequality has two downsides:

i) We first need to separately test that $a_x = b_x$

ii) We need 3 different measurement settings

– maybe with only 2 settings, everything can be modelled classically?

Consider setting with 2 measurements: $x = 0,1$; $y = 0,1$, again with outcomes $a_x, b_y = \pm 1$ (minimal setting).

Since $a_x = \pm 1$, $b_y = \pm 1$:

$$C = \underbrace{(a_0 + a_1)} b_0 + \underbrace{(a_0 - a_1)} b_1 = \pm 2$$

one of these must be 0, the other is ± 2.

average over many iterations (prob. distr.)

$$\Rightarrow |\langle C \rangle| \leq \langle |C| \rangle = 2$$

$$\boxed{|\langle a_0 b_0\rangle + \langle a_1 b_0\rangle + \langle a_0 b_1\rangle - \langle a_1 b_1\rangle| \leq 2}$$

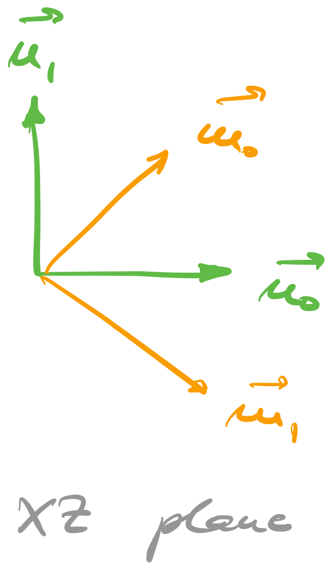
"CHSH inequality" (Clauser, Horne, Shimony, Holt)

Violation of CHSH inequality in quantum theory:

Take $|\psi\rangle = |\psi^-\rangle$

$a_x \longleftrightarrow \vec{u}_x \cdot \vec{\sigma}^A$

$b_y \longleftrightarrow \vec{w}_y \cdot \vec{\sigma}^B$



Have seen:

$$\langle a_x b_y\rangle = -\cos\theta$$

$$\Longrightarrow \langle a_0 b_0\rangle = \langle a_0 b_1\rangle = \langle a_1 b_0\rangle = -\frac{1}{\sqrt{2}}$$

$$\langle a_1 b_1\rangle = +\frac{1}{\sqrt{2}}$$

$$\Longrightarrow |\langle a_0 b_0\rangle + \langle a_1 b_0\rangle + \langle a_0 b_1\rangle - \langle a_1 b_1\rangle| = 2\sqrt{2} > 2 \;!$$

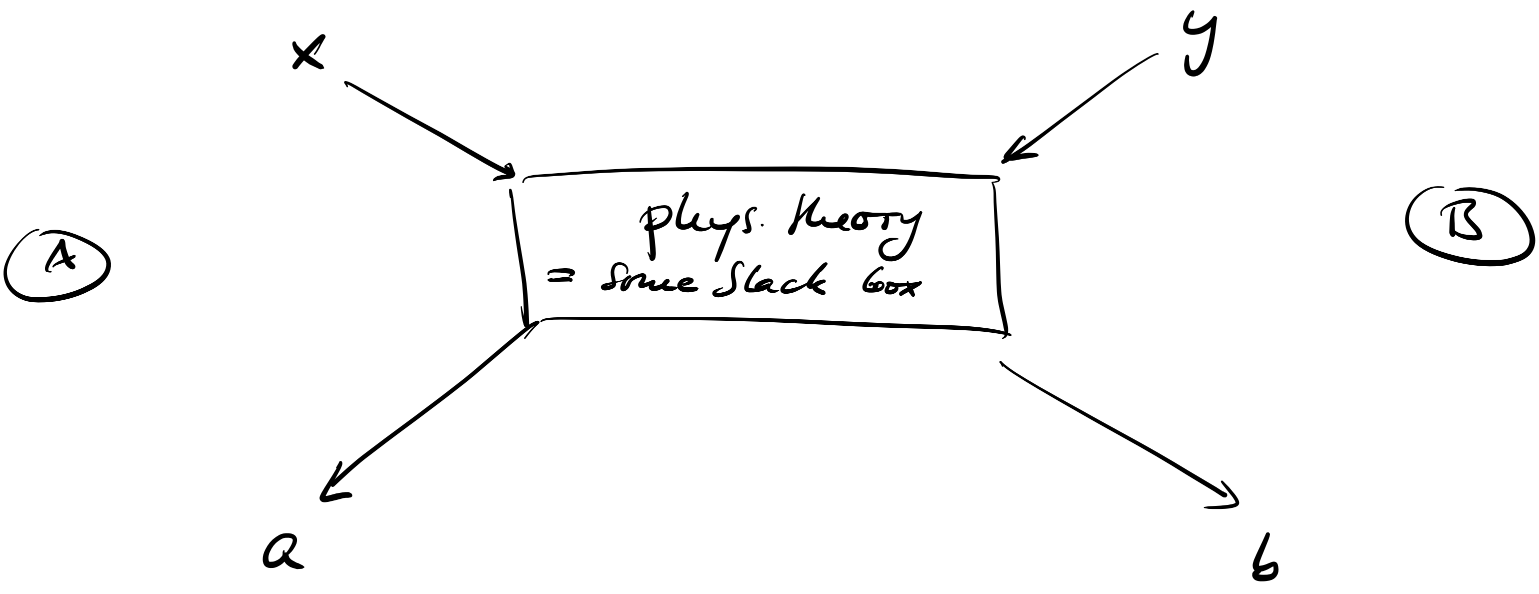
CHSH inequality violated!

Note: This violation is optimal (maximal) within QM.

(But: With a general local theory, $|\langle C \rangle| = 4$ can be obtained: QM is more restrictive than general local theories.)

c) Formal setup and local hidden variable theories

Formal setup for physical theories in bipartite setting:



A: input x (meas. setting),
output a (meas. result)

B: input y, output b.

Any physical theory is characterized by a conditional probability distribution

$$P(a,b|x,y)$$

to obtain a & b given x & y, where

$$\sum_{a,b} P(a,b|x,y) = 1 \quad \forall x,y.$$

<u>Question</u>: Which $P(a,b|x,y)$ are consistent with a given physical theory?

Classical physics:

"<u>Local hidden-variable (LHV) model</u>":

All outcomes are pre-determined by some "hidden" variable λ, which is chosen according to some distribution & given to A & B, who act independently (i.e., locally) conditioned on λ.

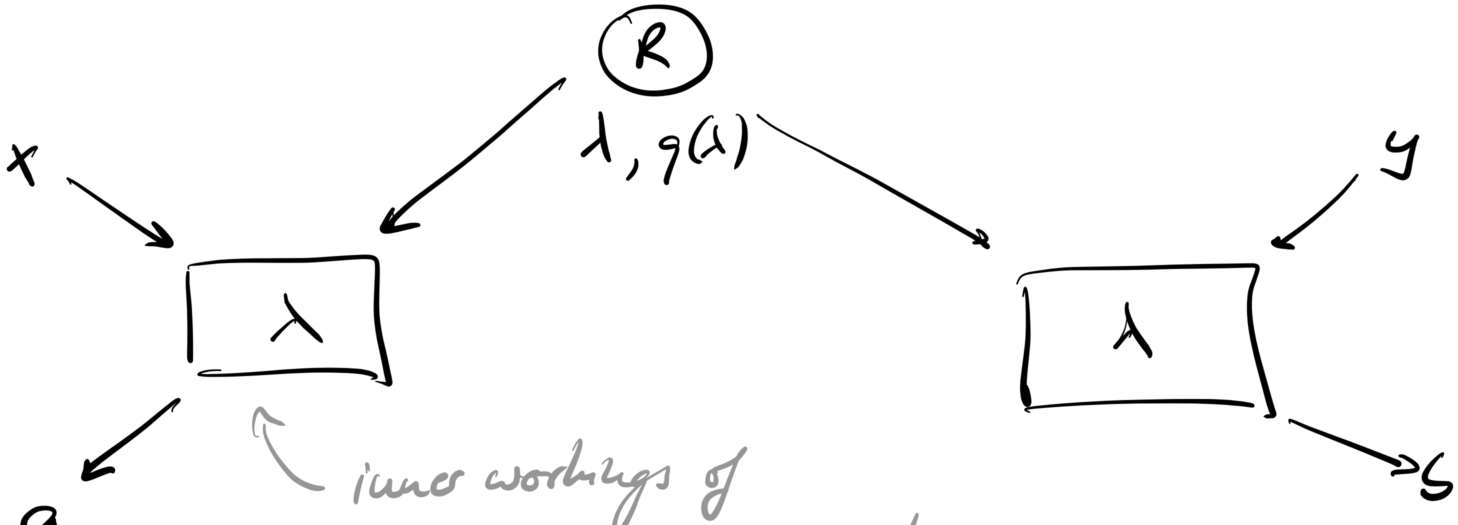
("<u>Local realism</u>": Outcomes exist indep. of meas. -realism- and no (faster-than-light) communication between A & B — locality.)

i.e.:

$$(\circledast) \quad P(a,b|x,y) = \sum_{\lambda} q(\lambda)\, P^A_\lambda(a|x)\, P^B_\lambda(b|y)$$

prob. distr. over λ

can be made deterministic by putting all randomness in λ & $q(\lambda)$.



inner workings of box depend <u>only on λ</u>!

How have we been using the LHV form $\circledast$ above in the derivation of Bell-type inequalities?

E.g. CHSH:

$$\langle C\rangle = \langle a_0 b_0\rangle + \langle a_1 b_0\rangle + \langle a_0 b_1\rangle - \langle a_1 b_1\rangle$$

$$= \sum_{x,y} (-1)^{xy} \langle a_x b_y\rangle$$

$$= \sum_{x,y} (-1)^{xy} \left[\sum_{a_x, b_y} a_x b_y \; P(a_x, b_y | x, y) \right]$$

$$= \sum q(\lambda)\, P^A_\lambda(a_x|x)\, P^B_\lambda(b_y|y)$$

$$= \sum_\lambda q(\lambda) \underbrace{\sum_{x,y} (-1)^{xy} \left[\sum_{a_x, b_y} a_x b_y \; P^A_\lambda(a_x|x)\, P^B_\lambda(b_y|y) \right]}_{=:\, E_\lambda}$$

Then, $|\langle C\rangle| \leq \sum_\lambda q(\lambda)\, |E_\lambda|$, and

$$|E_\lambda| = \left| \sum_{x,y} (-1)^{xy} \underbrace{\left(\sum_{a_x} a_x \; P^A_\lambda(a_x|x) \right)}_{= \langle a_x\rangle_\lambda} \underbrace{\left(\sum_{b_y} b_y \; P^B_\lambda(b_y|y) \right)}_{= \langle b_y\rangle_\lambda} \right|$$

$$= \left| \left(\langle a_0\rangle_\lambda + \langle a_1\rangle_\lambda \right) \langle b_0\rangle_\lambda + \left(\langle a_0\rangle_\lambda - \langle a_1\rangle_\lambda \right) \langle b_1\rangle_\lambda \right|$$

$$\leq \left| \langle a_0 \rangle_\lambda + \langle a_1 \rangle_\lambda \right| \underbrace{\left| \langle b_0 \rangle_\lambda \right|}_{\leq 1} + \left| \langle a_0 \rangle_\lambda - \langle a_1 \rangle_\lambda \right| \underbrace{\left| \langle b_1 \rangle_\lambda \right|}_{\leq 1}$$

$$\leq \left| \langle a_0 \rangle_\lambda + \langle a_1 \rangle_\lambda \right| + \left| \langle a_0 \rangle_\lambda - \langle a_1 \rangle_\lambda \right|$$

$$\leq 2 \max \left\{ \left| \langle a_0 \rangle_\lambda \right|, \left| \langle a_1 \rangle_\lambda \right| \right\} \leq 2.$$

Where have we used the LHV condition $\circledast$?

→ In factorizing

$$\langle a_x b_y \rangle_\lambda = \sum_{a_x, b_y} a_x b_y \; P(a_x, b_y | x, y)$$

$$= \left(\sum_{a_x} a_x \; P_\lambda^A(a_x | x) \right) \left(\sum_{b_y} b_y \; P_\lambda^B(b_y | y) \right)$$

$$= \langle a_x \rangle_\lambda \langle b_y \rangle_\lambda$$

— otherwise, it does not make sense to talk about $\langle a_x \rangle$ indep. of the value of y, and $\langle a_0 b_0 + a_0 b_1 + a_1 b_0 - a_1 b_1 \rangle$ cannot be factorized (either as expectation values $\langle a_i \rangle \langle b_j \rangle$, or classical variables — cons — which is in essence the same.)