4. Entanglement conversion & quantification

a) Introduction and setup

Entanglement = those properties of a system which cannot be changed using **Local Operations and Classical Communication (LOCC)**.

(Note: This is typically the **definition** of entanglement.)

Question: When – and how – can we convert entangled states into another by LOCC?

Relevance:

- Different protocols might require different – "cheaper" or "more expensive" – entangled states.
- We can easily produce some entangled state but need a different one.
- Can be used to <u>quantify</u> entanglement via conversion rate, e.g. to some "gold standard"

– for instance, how many "e-bits" $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ are contained in a state? (And is there a meaning to "$|\chi\rangle$ contains 0.7 e-bits?")

Let us first consider pure states!

Known from Chapter II:

Same Schmidt coefficients $\Longleftrightarrow$ states can be converted by local unitaries (subclass of LOCC!)

Question: What if Schmidt coefficients are different?

Example: $|\chi\rangle = \sqrt{\frac{2}{3}}|00\rangle + \sqrt{\frac{1}{3}}|11\rangle$

$|\phi^+\rangle = \sqrt{\frac{1}{2}}|00\rangle + \sqrt{\frac{1}{2}}|11\rangle$

Question 1: Can we do $|\phi^+\rangle \xrightarrow{\text{LOCC}} |\chi\rangle$?

convert using LOCC.

A does POVM $\{\Pi_0, \Pi_1\}$,

$$\Pi_0 = \begin{pmatrix} \sqrt{2/3} & \\ & \sqrt{1/3} \end{pmatrix}; \quad \Pi_1 = \begin{pmatrix} \sqrt{1/3} & \\ & \sqrt{2/3} \end{pmatrix}.$$

Post-measurement states $|\tilde{\psi}_k\rangle = (\Pi_k \otimes I)|\phi^+\rangle$:

$$|\tilde{\psi}_0\rangle = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{2}{3}}|00\rangle + \sqrt{\frac{1}{3}}|11\rangle\right)$$

$$|\tilde{\psi}_1\rangle = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{1}{3}}|00\rangle + \sqrt{\frac{2}{3}}|11\rangle\right)$$

$\Rightarrow$
- $p_0 = 1/2$: $|\psi_0\rangle = \sqrt{\frac{2}{3}}|00\rangle + \sqrt{\frac{1}{3}}|11\rangle = |\chi\rangle$ ✓ SUCCESS!
- $p_1 = 1/2$: $|\psi_1\rangle = \sqrt{\frac{1}{3}}|00\rangle + \sqrt{\frac{2}{3}}|11\rangle$

→ wrong state, but right Schmidt coefficients → can be fixed by local unitaries! (specifically, A & B need to apply $X \otimes X$).

Protocol for conversion $|\phi^+\rangle \xrightarrow{\text{LOCC}} |\chi\rangle$:

① A does POVM $\{\Pi_0, \Pi_1\}$

② A sends her outcome to B

③ If result is 1, both apply X.

⟹ Success probability $p = p_0 + p_1 = 1$!

Note: This is the best possible, since POVM cannot increase Schmidt rank → we cannot get more than one copy of $|\phi^+\rangle$ with any prob.!

Question 2: Can we do $|\chi\rangle \xrightarrow{\text{LOCC}} |\phi^+\rangle$?

A does POVM $\{\Pi_0, \Pi_1\}$,

$$\Pi_0 = \begin{pmatrix} \sqrt{1/2} & \\ & 1 \end{pmatrix}, \quad \Pi_1 = \begin{pmatrix} \sqrt{1/2} & \\ & 0 \end{pmatrix}$$

→ $|\tilde{\psi_0}\rangle = \sqrt{1/3}\,|00\rangle + \sqrt{1/3}\,|11\rangle$

$|\tilde{\psi_1}\rangle = \sqrt{1/3}\,|00\rangle$.

Outcome:

- $p_0 = \frac{2}{3}$: $|\psi_0\rangle = |\phi^+\rangle$ ✓ SUCCESS
- $p_1 = \frac{1}{3}$: $|\psi_1\rangle = |00\rangle$ ✗ FAILURE

(no ent. left → cannot salvage state!)

Protocol for $|X\rangle \xrightarrow{\text{LOCC}} |\phi^+\rangle$

① A does POVM $\{\Pi_0, \Pi_1\}$.

② A sends result to B.

③ If result is 0, they have obtained $|\phi^+\rangle$.
Otherwise, they declare failure.

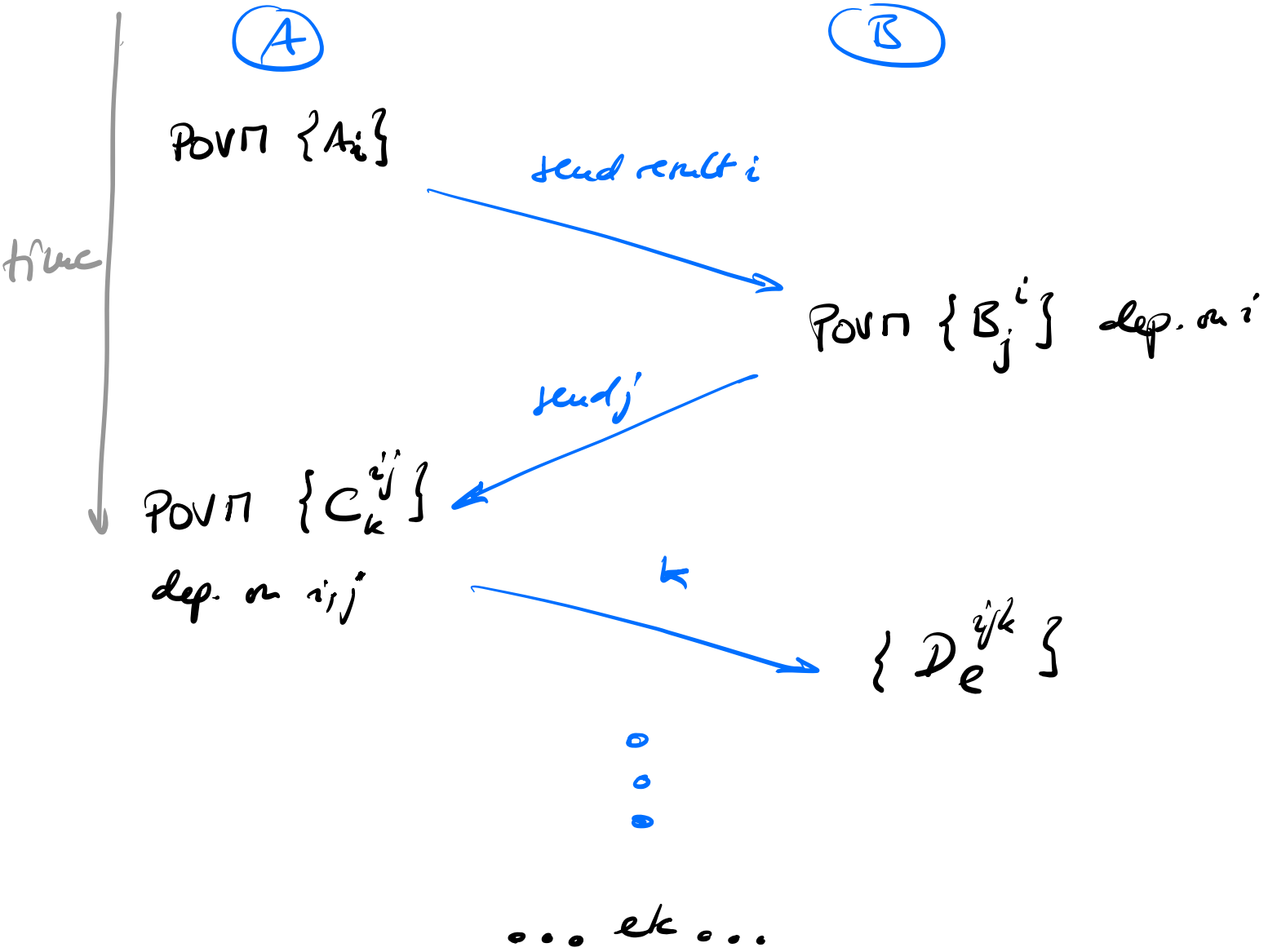
⟹ Success probability $p = p_0 = \frac{2}{3}$.

Note: Will see: This is indeed the best conversion rate possible.

Big drawback: Conversion is not reversible — we cannot use it to attach one number to an entangled state (i.e. quantify entanglement).

5) The most general LOCC protocol

What is the most general LOCC protocol?



... etc ...

this can go any number of rounds!

Total map implemented:

$$\rho \mapsto \sum (\dots \cdot C_k^{ij} \cdot A_i) \otimes (\dots \cdot D_e^{ijk} B_j^i)\, \rho\, (\,)^\dagger \otimes (\,)^\dagger$$

Very complicated structure!

(limit/closure over $N \to \infty$ rounds, ...)

Can we simplify this?

For general mixed states ρ_{AB} — no!

But: For LOCC protocols acting on pure states $|\psi\rangle_{AB}$, a significant simplification is possible:

Theorem: For a pure state $|\psi\rangle$, any LOCC protocol above can be replaced by the following protocol:

POVM $\{\Pi_k\}$ — send k → apply unitary U_k

$$|\psi\rangle \longmapsto \Pi_k \otimes U_k |\psi\rangle,$$

where $\{\Pi_k\}$ is a POVM and the U_k are unitary.

That is: Alice does a POVM $\{\Pi_k\}$, sends the result to Bob, and Bob applies a unitary U_k.

This only requires one round, and only one-way classical communication!

Proof: → Homework.

(Idea: Any state $|\chi\rangle$ can be written as

$$|\chi\rangle = (\Pi \otimes I)|\phi^+\rangle = (I \otimes N)|\phi^+\rangle$$

→ A can use this to "simulate" any meas. of B through a diff. meas. on her side — if state is known!)

c) Single-copy conversion & majorization

Can we characterize the optimal single-copy conversion protocols for pure states?

Have seen: most general protocol:

$$|\psi\rangle \longmapsto |\tilde{\psi}_k\rangle = (M_k \otimes U_k)|\psi\rangle,$$

$$p_k = \| |\tilde{\psi}_k\rangle \|^2$$

with $\{M_k\}$ POVM, U_k unitary.

Converts $|\psi\rangle \longmapsto \{p_k, |\psi_k\rangle = \frac{1}{\sqrt{p_k}}|\tilde{\psi}_k\rangle\}$,

i.e. state $|\psi_k\rangle$ w/ prob. p_k.

To characterize conversion power of LOCC:

- Only Schmidt coefficients of initial state $|\psi\rangle$ and final states $|\psi_k\rangle$ relevant — anything else can be changed at any time by local unitaries.

- Schmidt coefficients = eigenvalues of reduced density matrices

$$\rho_A = \mathrm{tr}_B |\psi\rangle\langle\psi|$$

$$\rho_{A,k} = \mathrm{tr}_B |\psi_k\rangle\langle\psi_k|$$

- LOCC protocol acts naked as

$$\rho_A \longmapsto \{p_k, \rho_{A,k}\}$$

with $p_k \rho_{A,k} = \Pi_k \rho_A \Pi_k^\dagger$, $\sum \Pi_k^\dagger \Pi_k = \mathbb{I}$.

Rem: To characterize the power of LOCC protocols, we can – and will – equivalently study the following question:

Question: Given ρ_A and $\{p_k, \rho_{A,k}\}$, under which conditions does there exist a POVM Π_k, $\sum \Pi_k^\dagger \Pi_k = \mathbb{I}$, such that $p_k \rho_{A,k} = \Pi_k \rho_A \Pi_k^\dagger$,

or more generally $\{\Pi_{k,i_k}\}$, $\sum \Pi_{k,i_k}^\dagger \Pi_{k,i_k} = \mathbb{I}$,

such that $\rho_{A,k} = \sum_{i_k} \Pi_{k,i_k} \rho_A \Pi_{k,i_k}^\dagger$?

This allows for grouping of outcomes, such as e.g. for the example $|\phi^+\rangle \longrightarrow |\chi\rangle$

Definition: For $\lambda \in \mathbb{R}^d_{\geq 0}$, let $\lambda^\downarrow = (\lambda_1^\downarrow, \dots, \lambda_d^\downarrow)$, $\lambda_1^\downarrow \geq \lambda_2^\downarrow \geq \dots \geq 0$ denote the ordered version of λ.

Definition (Majorization): We say that λ is majorized by μ, or μ majorizes λ, denoted as

$$\lambda \prec \mu,$$

iff $\sum_{i=1}^k \lambda_i^\downarrow \leq \sum_{i=1}^k \mu_i^\downarrow \quad \forall\, k = 1, \dots, d,$

with equality for $k = d$.

Theorem: The following are equivalent

(i) $\lambda \prec \mu$

(ii) there exist permutations P_i and probabilities q_i, $\sum q_i = 1$, such that

$$\lambda = \sum q_i P_i \mu$$

(iii) there exists a doubly stochastic matrix Q

(i.e.: $Q_{ij} \geq 0$, $\sum_i Q_{ij} = 1 \ \forall j$, $\sum_j Q_{ij} = 1 \ \forall i$,

that is, Q describes a random process with the fully random distribution $\left(\frac{1}{d}, \dots, \frac{1}{d}\right)$ as a fixed point)

such that $\lambda = Q\mu$.

Intuitively, this is saying that when regarded as prob. distributions, λ is "more random" than μ, in the sense that it can be obtained from μ by adding in randomness.

Proof:

(ii) $\Rightarrow$ (iii): Let $Q = \sum q_i P_i$. ∎

(iii) $\Rightarrow$ (ii): This is known as **Birkhoff's Theorem**:

Every doubly stochastic Q is of the form $Q = \sum q_i P_i$. ∎

(ii) $\Rightarrow$ (i): First, note that for any λ,

$$\sum_{i=1}^{k} \lambda_i^{\downarrow} \geq \sum_{i=1}^{k} \lambda_{\pi(i)} \quad \text{for any perm. } \pi.$$

sum of k largest terms — sum of any k terms

Now let $\lambda := \sum q_s P_s \mu$. Then,

$$\sum_{i=1}^{k} \lambda_i^{\downarrow} = \sum_{i=1}^{k} \lambda_{\pi(i)} = \sum_{i=1}^{k} \sum_s q_s \left[P_s \mu\right]_{\pi(i)}$$

ordering = some perm. π

$$= \sum_s q_s \sum_{i=1}^{k} \mu_{P_s^{-1}(\pi(i))} \leq \sum_{i=1}^{k} \mu_i^{\downarrow}$$

$$\leq \sum_{i=1}^{k} \mu_i^{\downarrow}$$

∎

(i) $\Rightarrow$ (ii): Homework.

Idea: Find a permutation P s.th.

$$\lambda_1^{\downarrow} = \Big[\overbrace{\big(qI + (1-q)P\big)}^{Q} \mu^{\downarrow} \Big]_1 ,$$

then consider $(\lambda_2^{\downarrow}, \dots, \lambda_d^{\downarrow}) \prec ((Q\mu^{\downarrow})_2, \dots, (Q\mu^{\downarrow})_d)$,

& proceed by induction.

Remarks:

- Majorization defines a partial order on the space of prob. distributions.
- $\lambda \prec \mu$: λ "more disordered" than μ – in part., it has larger entropy! (Made rigorous by notion of "Schur concavity/convexity": for a concave/convex $f(x)$,

$$F(\lambda) = \sum f(\lambda_i) \text{ fulfills}$$

$$\lambda \prec \mu \Rightarrow F(\lambda) \overset{\geq}{(\leq)} F(\mu)$$

(Concave / Convex)

Majorization can be generalized to positive operators:

Definition: For $A, B \geq 0$, we define

$$A \prec B \quad :\Longleftrightarrow \quad \lambda^{\downarrow}(A) \prec \lambda^{\downarrow}(B),$$

with $\lambda^{\downarrow}(X)$ the ordered eigenvalues of X.

Theorem (Ky-Fan maximum principle):

For A hermitian,

$$\sum_{j=1}^{k} \lambda_j^{\downarrow}(A) = \max_{P} \operatorname{tr}(AP),$$

where the maximum is over all projectors P with rank k.

Proof:

"$\leq$": let $A = \sum_{j=1}^{d} \lambda_j^{\downarrow}(A) |q_j\rangle\langle q_j|$. Choose $\tilde{P} = \sum_{i=1}^{k} |q_j\rangle\langle q_j|$.

Then, $\sum_{j=1}^{k} \lambda_j^{\downarrow}(A) = \operatorname{tr}(A\tilde{P}) \leq \max_{P} \operatorname{tr}(AP)$.

"$\geq$": Write $P = \sum_{i=1}^{k} |p_i\rangle\langle p_i|$, with $\{|p_i\rangle\}_{i=1}^{d}$ ons.

Then, $\operatorname{tr}(AP) = \sum_{i=1}^{k} \langle p_i | A | p_i \rangle = \sum_{i=1}^{k} \sum_j |\langle p_i | q_j \rangle|^2 \lambda_j^{\downarrow}(A)$

$$= \sum_j \underbrace{\left(\sum_{i=1}^{k} |\langle p_i | q_j \rangle|^2 \right)}_{=: \omega_j} \lambda_j^{\downarrow}(A) = \circledast$$

Here, $0 \le \omega_j \le 1$, $\sum_j \omega_j = \sum_{i=1}^{k} \sum_j \langle p_i | \underbrace{q_j \rangle\langle q_j}_{\Sigma = I} | p_i \rangle = k$.

$$\circledast = \sum_j \omega_j \lambda_j^{\downarrow}(A) \le \sum_{j=1}^{k} \lambda_j^{\downarrow}(A)$$

(by re-distributing weight in ω_j towards the first k entries, which have largest $\lambda_j^{\downarrow}(A)$).

□

Corollary: $\lambda^{\downarrow}(A+B) \prec \lambda^{\downarrow}(A) + \lambda^{\downarrow}(B)$.

Proof: $\sum_{i=1}^{k} \lambda_i^{\downarrow}(A+B) = \max_{P;\, \operatorname{rank} P = k} \operatorname{tr}(P(A+B))$

$$\le \max_P \operatorname{tr}(PA) + \max_P \operatorname{tr}(PB)$$

$$= \sum_{i=1}^{k} \lambda_i^{\downarrow}(A) + \sum_{i=1}^{k} \lambda_i^{\downarrow}(B).$$

□

Theorem (single-copy entanglement conversion):

We can convert

$$|\psi\rangle \xrightarrow{\text{LOCC}} \{p_k, |\psi_k\rangle\}$$

by LOCC if and only if

$$\lambda^{\downarrow}(\rho) \prec \sum_{k=1}^{K} p_k \lambda^{\downarrow}(\rho_k),$$

where $\rho = \mathrm{tr}_A |\psi\rangle\langle\psi|$, $\rho_k = \mathrm{tr}_A |\psi_k\rangle\langle\psi_k|$.

↑ Eigenvalues of $\mathrm{tr}_B |\psi\rangle\langle\psi|$ and $\mathrm{tr}_B |\psi\rangle\langle\psi|$ are the same. But since Alice does POVM, tracing her will facilitate the proof.

Proof:

"$\Rightarrow$": A does some POVM $\{\Pi_k\}$. Then,

$$\sum p_k \rho_k = \rho \quad \text{(red. state of Bob!)}, \text{ and thus}$$

$$\sum_{k=1}^{K} p_k \lambda^{\downarrow}(\rho_k) = \sum_{k=1}^{K} \lambda^{\downarrow}(p_k \rho_k) \overset{\text{Corollary}}{\prec} \sum \lambda^{\downarrow}(\rho).$$

Note: This works regardless of whether there are zero outcomes – i.e., performs $\{\Pi_{k,i_k}\}$, where for a given k, all i_k, together with Bob's U_{k,i_k}, give the same $|\psi_k\rangle$ – since $\lambda^\downarrow(\rho_{k,i_k})$ is the same for all i_k (fixed k).

"$\Leftarrow$": $\lambda^\downarrow(\rho) \prec \sum_k p_k \lambda^\downarrow(\rho_k)$

$\Rightarrow \exists P_{j|k} q_j$ s.t. $\lambda^\downarrow(\rho) = \sum_{k,j} q_j P_{j|k} p_k \lambda^\downarrow(\rho_k)$

W.l.o.g., we can assume ρ, ρ_i are diagonal (i.e. $\rho = \mathrm{diag}(\lambda^\downarrow(\rho))$, etc.), since only the eigenvalues matter. — The corresponding POVMs $\{\Pi_k\}$ are related by unitaries, $\Pi_k \rightsquigarrow V_k \Pi_k W$.

Furthermore, if $\lambda^\downarrow(\rho)$ has zeros (i.e. zero eigenvalues), $\sum_k p_k \lambda^\downarrow(\rho_k)$ and thus each $\lambda^\downarrow(\rho_k)$ must have at least the same number of zeros (since $\lambda^\downarrow(\rho) \prec \sum p_k \lambda^\downarrow(\rho_k)$). We can then discard those zeros (i.e. POVM acts trivially on those).

Thus w.l.o.g.: ρ has full rank (invertible!).

Define POVM $\{\Pi_{kj}\}$ via $\Pi_{kj}\sqrt{\rho} = \sqrt{p_k q_j}\,\sqrt{\rho_k}\,P_j^\dagger$.

Then, $\sqrt{\rho}\left(\sum_{kj} M_{kj}^\dagger \Pi_{kj}\right)\sqrt{\rho} = \sum_{kj} p_k q_j P_j \rho_k P_j^\dagger = \rho$

($\rho_k = \mathrm{diag}(\lambda^k(p_1,\dots,p_d))$)

ρ full rk. $\Rightarrow \sum_{kj} \Pi_{kj}^\dagger \Pi_{kj} = I \Rightarrow \Pi_{kj}$ POVM!

Further, $\Pi_{kj}\rho\,\Pi_{kj}^\dagger = p_k q_j \rho_k \Rightarrow$ given k, outcomes for different j are all equal $\Rightarrow$ we can do them into one outcome, w/ outcome probability p_k.

$\Rightarrow$ by discarding all j-outcomes for fixed k, we obtain POVM for $\rho \longmapsto \{p_k, \rho_k\}$, i.e. $|\psi\rangle \mapsto \{p_k, |\psi_k\rangle\}$.

Example:

Optimal rate for $\left(\frac{1}{2}, \frac{1}{2}\right) \longleftrightarrow \left(\frac{2}{3}, \frac{1}{3}\right)$

vectors with Schmidt coefs.

- $\left(\frac{1}{2}, \frac{1}{2}\right) \prec \left(\frac{2}{3}, \frac{1}{3}\right)$: prob = 1 ✓

- $\left(\frac{2}{3}, \frac{1}{3}\right) \prec \frac{2}{3}\left(\frac{1}{2}, \frac{1}{2}\right) + \frac{1}{3}(1,0) = \left(\frac{2}{3}, \frac{1}{3}\right)$

max. possible value.

prob = 2/3 ✓

- we also see that Schmidt rank cannot be increased, e.g.: Can we do

$$|\phi^+\rangle \longrightarrow \left(\sqrt{\tfrac{2}{3}}|00\rangle + \sqrt{\tfrac{1}{3}}|11\rangle\right)^{\otimes 2} \; ?$$

$$\left(\tfrac{1}{2}, \tfrac{1}{2}, 0, 0\right) \not\prec p\left(\tfrac{4}{9}, \tfrac{2}{9}, \tfrac{2}{9}, \tfrac{1}{9}\right) + (1-p)\cdot \text{Shift}$$

impossible!

d) Asymptotic protocols

Single-copy entanglement conversion:

most reversible!

E.g.: $|\phi^+\rangle \rightleftharpoons |\chi\rangle = \sqrt{\frac{2}{3}}|00\rangle + \sqrt{\frac{1}{3}}|11\rangle$

need 1 ebit for "$\longrightarrow$",

get $\frac{1}{3}$ ebit from "$\longleftarrow$".

→ Entanglement is lost

→ need at least two numbers to quantify entanglement:

ebits needed to build state

ebits extractable from state

Can we do better if we work with more copies?

$|\chi\rangle^{\otimes 2} \longrightarrow p_2 |\phi^+\rangle^{\otimes 2} + p_1 |\phi^+\rangle^{\otimes 1}$?

$|\chi\rangle^{\otimes 3} \longrightarrow p_3 |\phi^+\rangle^{\otimes 3} + p_2 |\phi^+\rangle^{\otimes 2} + \ldots$?

Average yield of max. ent. states $|\phi^+\rangle$

= "# of ebits per copy of $|\chi\rangle$":

$$\bar{P} = \frac{P_1 + 2P_2 + 3P_3 + \dots}{k}$$

$k \leftarrow$ # of copies.

E.g. for $|\chi\rangle^{\otimes 3}$:

$$\left(\frac{8}{27}, \frac{4}{27}, \frac{4}{27}, \frac{4}{27}, \frac{2}{27}, \frac{2}{27}, \frac{2}{27}, \frac{1}{27}\right) \prec$$

$$P_3\left(\tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}\right) +$$

$$P_2\left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, 0, 0, 0, 0\right) +$$

$$P_1\left(\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, 0, 0, 0, 0\right) +$$

$$P_0\,(1, 0, 0, 0, 0, 0, 0, 0)$$

$$\| \quad 1-(P_1+P_2+P_3)$$

$p_3 = 8/27,\ p_2 = 16/27,\ p_1 = 0,\ p_0 = 3/27,$

gives a valid solution.

$$\bar{p} = \frac{3 \cdot \frac{8}{27} + 2 \cdot \frac{16}{27}}{3} = \frac{56}{81} > 2/3 !$$

→ Improvement over single-copy protocols!

How good can we get by using $N \to \infty$ copies?

Requirements for asymptotic protocols

- Convert $|\phi^+\rangle^{\otimes n} \leftrightarrows |X\rangle^{\otimes N}$ with a rate $R = \lim \frac{n}{N}$ as $N, n \to \infty$

  (we aim for best rate R: large R for "←", small R for "→")

- Need not be deterministic:

  Success probability $p \to 1$ as $N, n \to \infty$

- Need not be perfect:

  Require that distance δ from correct state goes to zero, $\delta \to 0$, as $N, n \to \infty$.

(Note: We can afford these imperfections since asymptotically, they vanish: Not meaningful in finite-copy scenario.)

How can we measure error δ?

Definition (Fidelity): For two states $|\psi\rangle$ and $|\phi\rangle$,

$F = |\langle\psi|\phi\rangle|^2$ is called the *fidelity* of $|\psi\rangle$ and $|\phi\rangle$.

Lemma: $\delta := 1 - F$ bounds the error in any expectation value,

$$|\langle\psi|O|\psi\rangle - \langle\phi|O|\phi\rangle| \leq 2\sqrt{\delta}\,\|O\|_\infty,$$

with $\|O\|_\infty = \sup \frac{\|O|\psi\rangle\|}{\||\psi\rangle\|}$ the *operator norm*.

That is: δ bounds how well we can (or cannot) distinguish $|\phi\rangle$ and $|\psi\rangle$ by <u>any physical test</u>!

<u>Proof</u>: → Homework.

We use $\delta = 1 - F$ to measure the error, and we require a good <u>global</u> distance, i.e.

$$|\phi^+\rangle^{\otimes M} \longrightarrow |\Theta_N\rangle \approx |\chi\rangle^{\otimes N}$$

$$\uparrow$$

$$|\langle a_N | (|\chi\rangle^{\otimes N})|^2 \longrightarrow 1$$

(and vice versa).

Let us now consider some state

$$|\chi\rangle = \sum_{x=1}^{d} \sqrt{p(x)}\ |x\rangle_A |x\rangle_B .$$

Then, $|\chi\rangle^{\otimes N} = \sum_{x_1,\dots,x_N} \sqrt{p(x_1)\cdot\dots\cdot p(x_N)}\ |x_1,\dots,x_N\rangle_A |x_1,\dots,x_N\rangle_B$,

i.e., the probabilities $p(x_1, \dots, x_N)$ associated to a configuration $x_1, \dots, x_N$ describe independently & identically distributed (iid) random variables with prob. $p(x_i)$.

Theorem (Law of large numbers (LLN))

Let y be a random variable with prob. $p(y)$. Then,

$$\forall \varepsilon > 0 \;\; \forall \delta > 0 \;\; \exists N_0 \;\; \forall N \geq N_0$$

$$\text{prob}\left(\left| \frac{1}{N} \sum_{i=1}^{N} y_i - E(y) \right| > \varepsilon \right) < \delta$$

where $E(y) = \sum_y p(y)\, y$.

(Proof $\rightarrow$ probability theory.)

How are the outcomes of an iid. source distributed?

What is a *typical outcome* of an iid. source?

Example: $x = 0, 1$; $p_0 = p$; $p_1 = 1-p$.

k times outcome 0, $N-k$ times outcome 1:

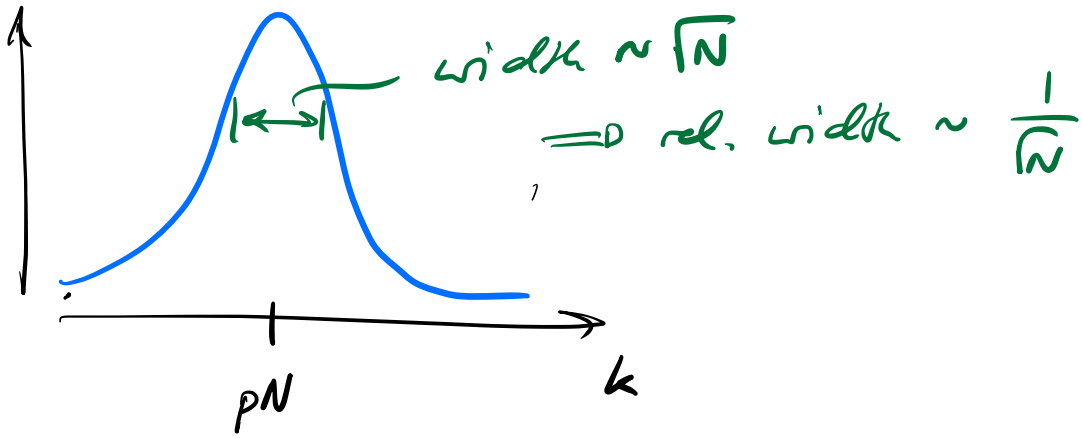
$$\text{Prob.} = p^k (1-p)^{N-k}$$

possibilities: $\binom{N}{k}$

$\Rightarrow$ Binomial distribution:

total prob. for $k \times 0$, $(N-k) \times 1$:

$$p^k (1-p)^{N-k} \binom{N}{k}$$



– i.e., for large N, we expect that $\frac{k}{N} \approx p$ with high probability $\to 1$ as $N \to \infty$.

General case:

"Typical" output: Expect output x $N \cdot p(x)$ times.

$$\Rightarrow p(x_1, \dots, x_N) = p(x_1) \cdot \dots \cdot p(x_N) \approx p(1)^{Np(1)} \cdot \dots \cdot p(d)^{Np(d)}$$

$$\Rightarrow -\log p(x_1, \dots, x_N) \approx N \cdot \underbrace{\left(- \sum_{x=1}^{d} p(x) \log p(x) \right)}_{=: H(p)}$$

$H(p)$: Shannon entropy of p.

our logs are base 2!

$\Rightarrow$ we typically expect to see sequences $x_1, \dots, x_N$ for which

$$p(x_1, \dots, x_N) \approx 2^{-NH(p)},$$

and there are $\approx 2^{NH(p)}$ such typical sequences.

More formally, this is defined as follows:

Definition (ε-typical sequence):

For an iid-variable x, we say that $x_1, \dots, x_N$ is an **ε-typical sequence** if

$$2^{-N(H(p)+\varepsilon)} \leq p(x_1, \dots, x_N) \leq 2^{-N(H(p)-\varepsilon)}$$

We denote the set of ε-typical sequences by $T(N, \varepsilon)$.

Theorem:

$\forall \varepsilon > 0 \; \forall \delta > 0 \; \exists N_0 \; \forall N \geq N_0$ such that

1) a random sequence $x_1, \dots, x_N$ of length N is ε-typical w/ prob. $\geq 1-\delta$.

2) $(1-\delta)\, 2^{N(H(p)-\varepsilon)} \leq |T(N, \varepsilon)| \leq 2^{N(H(p)+\varepsilon)}$

Proof: $-\log p(x_i)$ is iid variable (w/ prob. $p(x_i)$ for outcome i)

$\overset{LLN}{\Longrightarrow} \forall \varepsilon, \delta \; \exists N_0 \; \forall N \geq N_0$:

$$\text{prob}\left(\left| \underbrace{\frac{1}{N} \sum_{i=1}^{N} -\log p(x_i)}_{= -\log p(x_1, \dots, x_N)} - \underbrace{E\left(-\log p(x)\right)}_{= \sum_x p(x)\left(-\log p(x)\right) = H(p)} \right| > \varepsilon \right) < \delta$$

$$\Rightarrow \text{prob}\left(\left| -\frac{1}{N} \log p(x_1, \dots, x_N) - H(p) \right| > \varepsilon \right) < \delta$$

1) $\Rightarrow$ w. prob. $\geq 1-\delta$,

$$-N(H(p)+\varepsilon) \leq \log p(x_1, \dots, x_N) \leq -N(H(p)-\varepsilon) \quad \blacksquare$$

2) $$1 \geq \sum_{x_1, \dots, x_N \in T(N,\varepsilon)} p(x_1, \dots, x_N)$$

$$\overset{\text{Def. } T(N,\varepsilon)}{\geq} \sum_{T(N,\varepsilon)} 2^{-N(H(p)+\varepsilon)}$$

$$= |T(N,\varepsilon)| \cdot 2^{-N(H(p)+\varepsilon)}$$

$$1-\delta \leq \sum_{T(N,\varepsilon)} p(x_1,\dots,x_N) \leq |T(N,\varepsilon)| \cdot 2^{-N(H(p)-\varepsilon)}$$

In words:

ε-typical sequence: $\Leftrightarrow$ $\frac{\log p(x_1,\dots,x_N)}{N}$ ε-close to $H(p)$

Asymptotically, a sequence is ε-typical w/ prob $\to 1$, and there are $\sim 2^{NH(p)}$ ε-typical sequences.

Note: Typical sequences are an important concept in classical information theory ($\to$ data compression etc.!)

Application of typicality to entanglement concentration:

$$|\chi\rangle = \sum_x \sqrt{p(x)}\, |x\rangle_A |x\rangle_B$$

$$\Rightarrow |\chi\rangle^{\otimes N} = \sum_{x_1,\dots,x_N} \sqrt{p(x_1)\cdots p(x_N)}\; |x_1,\dots,x_N\rangle_A |x_1\dots x_N\rangle_B$$

Fix some $\varepsilon > 0$ and $\delta > 0$, and a matching N_0. Let

$$|\tilde{\vartheta}_N\rangle := \sum_{x_1,\dots,x_N \in T(N,\varepsilon)} \sqrt{p(x_1)\cdot\ldots\cdot p(x_N)}\, |x_1,\dots,x_N\rangle |x_1,\dots,x_N\rangle ,$$

$$|\vartheta_N\rangle := \frac{|\tilde{\vartheta}_N\rangle}{\sqrt{\langle\tilde{\vartheta}_N|\tilde{\vartheta}_N\rangle}}$$

We have (for $N \geq N_0$):

$$\langle\vartheta_N|\left(|X\rangle^{\otimes N}\right) = \frac{\sum_{T(N,\varepsilon)} p(x_1,\dots,x_N)}{\sqrt{\sum_{T(N,\varepsilon)} p(x_1,\dots,x_N)}} \geq \sqrt{1-\delta} \geq 1-\delta$$

$\geq 1-\delta$ for $N \geq N_0$

and $|T(N,\varepsilon)| \leq 2^{N(H(p)+\varepsilon)}$.

→ That is: Instead of converting $|\phi^+\rangle^{\otimes n} \rightleftarrows |X\rangle^{\otimes N}$,

we can instead convert $|\phi^+\rangle^{\otimes n} \rightleftarrows |\vartheta_N\rangle$,

as the error δ can be taken to zero.

Protocol for $|\phi^+\rangle^{\otimes n} \longrightarrow |\vartheta_N\rangle \approx |\chi\rangle^{\otimes N}$:

A prepares $|\vartheta_N\rangle$ locally & teleports Bob's part to Bob

$\Rightarrow$ uses $n = \log |T(N,\varepsilon)| \leq N(H(\rho)+\varepsilon)$ copies of $|\phi^+\rangle$.

$\lim \frac{n}{N} = H(\rho) + \varepsilon$ "entanglement dilution rate"

Since any $\varepsilon > 0$ admissible: Can achieve

asymptotic rate $R_{dilute} = H(\rho)$ for ent. dilution.

(Alternatively: use majorization, same result).

Protocol for $|\chi\rangle^{\otimes N} \approx |\vartheta_N\rangle \longrightarrow |\phi^+\rangle^{\otimes n}$:

- Can consider $|\vartheta_N\rangle$ as fidelity $\to 1$.
- $|\tilde{\vartheta}_N\rangle$: $|T(N,\varepsilon)|$ Schmidt coefficients,

  largest Schmidt coeff. $\leq 2^{-N(H(\rho)-\varepsilon)}$

- $\Rightarrow |\vartheta_N\rangle$: largest Schmidt coeff.

  $\leq \frac{1}{1-\delta} 2^{-N(H(\rho)-\varepsilon)}$

Choose m s.t. $\frac{1}{1-\delta} 2^{-N(H(\rho)-\varepsilon)} \leq 2^{-m}$

$\Rightarrow (\underbrace{2^{-m}, 2^{-m}, \ldots, 2^{-m}}_{2^m \text{ times}}, 0, \ldots 0) \succ (\text{Schmidt coef. of } |\vartheta_N\rangle)$

$\Rightarrow$ can convert $|\vartheta_N\rangle$ to $|\phi^+\rangle^{\otimes m}$ by LOCC.

Protocol: i) A projects onto ε-typ. subspace

$\longrightarrow$ obtains $|\vartheta_N\rangle$.

(i.e.: POVM $\{\Pi_0 = \Pi_{\varepsilon\text{-typ}}, \Pi_1 = \mathbb{I} - \Pi_0\}$)

ii) A & B convert $|\vartheta_N\rangle$ to $|\phi^+\rangle^{\otimes m}$.

Protocol works for any $m \leq N(H(\rho)-\varepsilon) + \log(1-\delta)$

$\Rightarrow$ Rate $R = \lim \frac{m}{N} \longrightarrow H(\rho) - \varepsilon \quad \forall \varepsilon > 0.$

$\Rightarrow$ Asymptotic rate $R_{dist} = H(\rho)$ for entanglement distillation.

We find: Asymptotically (i.e., $N, m \to \infty$),

distillation rate = dilution rate = $H(\rho)$

Is this optimal?

→ Yes! Otherwise, we could "go in circles" and increase # of $|\phi^+\rangle$ in every iteration by LOCC! ↯

Note: Instead of the Shannon entropy $H(p)$, we typically use the

$$\boxed{\text{von Neumann entropy} \quad S(\rho) = -\operatorname{tr}(\rho \log \rho)}$$

- since $\log\rho$ is defined in the eigenbasis, we have

$$-\operatorname{tr}\rho\log\rho = -\sum p_i \log p_i \quad \text{with } (p_i) = \operatorname{eig}(\rho),$$

i.e., $H(p) = S(\operatorname{tr}_B |\psi\rangle\langle\psi|) = S(\operatorname{tr}_A |\psi\rangle\langle\psi|)$.

Note: This protocol allows us to go between any two states $|\psi\rangle^{\otimes N}$ and $|\phi\rangle^{\otimes M}$ asymptotically reversibly, provided

$$\boxed{N \cdot S(\operatorname{tr}_B |\psi\rangle\langle\psi|) = M \cdot S(\operatorname{tr}_B |\phi\rangle\langle\phi|)}$$

(by going via $|\phi^+\rangle^{\otimes K}$)

$S(\mathrm{tr}_B |\psi\rangle\langle\psi|) =: E(|\psi\rangle)$ can thus be seen as a unique measure of entanglement (in an asymptotic LOCC setting).

Key result: The entropy of entanglement

$$E(|\psi\rangle) = S(\mathrm{tr}_A |\psi\rangle\langle\psi|) = S(\mathrm{tr}_B |\psi\rangle\langle\psi|)$$

uniquely quantifies the amount of entanglement in a pure bipartite state in an asymptotic setting.