5. Entanglement of mixed states

a) Introduction

When is a mixed bipartite state ρ_{AB} entangled?

Different possible definitions:

(i) If ρ_{AB} cannot be created by LOCC.

(ii) If we can extract ebits $|\phi^+\rangle$ from ρ_{AB}.

(iii) If it helps us to do some task better in an LOCC setting (it is a "resource").

... any of those could be in a finite-copy or asymptotic setting!

Clearly, (ii) $\geq$ (iii) $\geq$ (i)

"$\geq$": is a stronger condition, i.e. satisfied by less states.

We use the weakest notion (ii) to define entangled states.

Definition:

A bipartite state ρ_{AB} is called separable if it can be written as

$$(*) \quad \boxed{\rho_{AB} = \sum p_i \, \rho_i^A \otimes \rho_i^B} \quad \text{"separable state"}$$

for some $p_i \geq 0$, $\sum p_i = 1$;

$\rho_i^A \geq 0$, $\rho_i^B \geq 0$; $\operatorname{tr} \rho_i^A = \operatorname{tr} \rho_i^B = 1$

— i.e., ρ_{AB} can be prepared by LOCC.

If ρ_{AB} is not separable — i.e., it has no decomposition of the form $(*)$ — it is called entangled.

Given a state ρ_{AB}, how can we tell if it is entangled?

Problem: Given ρ_{AB}, need to check all decompositions

$$\rho_{AB} = \sum p_i \, \rho_i^{AB} = \sum p_i \, \rho_i^A \otimes \rho_i^B$$

(or $\rho_{AB} = \sum p_i |\psi_i^{AB}\rangle\langle\psi_i^{AB}|$) to find if there is a separable one. (Ambiguity in ensemble decomposition is an isometry, i.e., we need to optimize over isometries!)

$\rightarrow$ Difficult!

(In fact, the general problem has been shown to be NP-hard in the dimension of the space.)

$\rightarrow$ Need partial solutions, e.g. ways to certify a given state is entangled.

b) Entanglement witnesses

Structure of the set $\mathcal{S}$ of separable states:

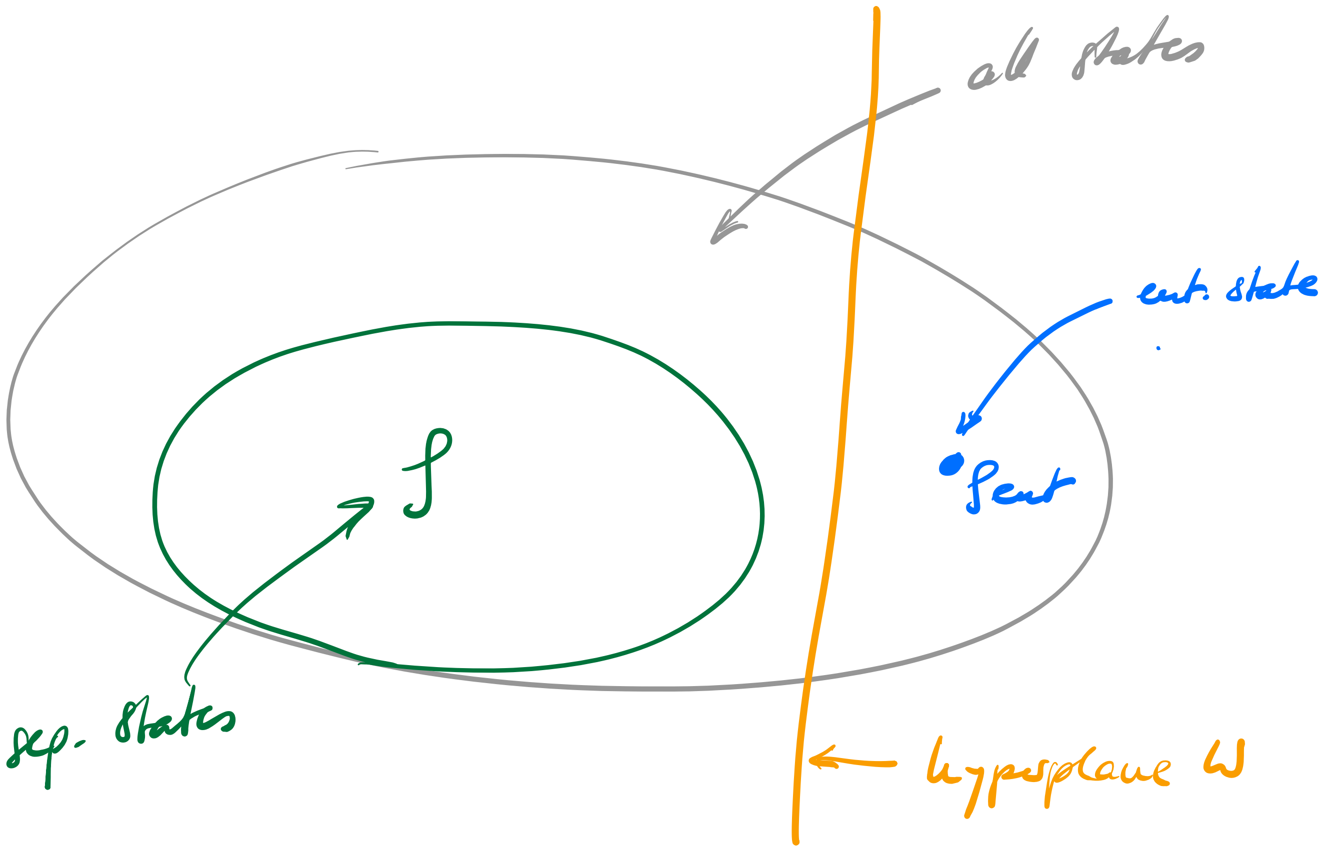
Let $\rho = \sum_{i=1}^{N} p_i \rho_i^A \otimes \rho_i^B \in \mathcal{S}$; $\sigma = \sum_{j=1}^{M} q_j \sigma_j^A \otimes \sigma_j^B \in \mathcal{S}$

$\Rightarrow$ for $\lambda \in [0;1]$,

$$\underline{\lambda\rho + (1-\lambda)\sigma} = \lambda p_1 \rho_1^A \otimes \rho_1^B + \ldots + \lambda p_N \rho_N^A \otimes \rho_N^B + \ldots$$

$$+ (1-\lambda) q_1 \sigma_1^A \otimes \sigma_1^B + \ldots + (1-\lambda) q_M \sigma_M^A \otimes \sigma_M^B \underline{\in \mathcal{S}}$$

$\Rightarrow$ the set $\mathcal{S}$ of separable states forms a <u>convex set</u>



For any state $\rho_{ent} \notin \mathcal{S}$, we can find a <u>hyperplane</u> which separates ρ_{ent} from $\mathcal{S}$.

More generally, we can construct hyperplanes s.th. all points on one side of the plane are entangled (but not the other way).

Any hyperplane is of the form $tr[X\rho] + c = 0$,

i.e., $tr[\underbrace{(X + cI)}_{=:W}\rho] = tr[W\rho] = 0.$

We can choose $W = W^\dagger$ as we work in the space of hermitian matrices.

Then:

$$tr(\rho W) > 0 : \rho \text{ left of hyperplane } W$$

$$tr(\rho W) < 0 : \rho \text{ right of hyperplane } W$$

I.e. for a hyperplane as above:

$$\rho \text{ separable} \Longrightarrow tr(W\rho) \geq 0$$

and thus

$$tr(W\rho) < 0 \Longrightarrow \rho \text{ entangled.}$$

Definition (entanglement witness):

An operator $W = W^\dagger$ such that

$$\rho \text{ separable} \implies \mathrm{tr}(W\rho) \geq 0$$

is called an entanglement witness.

Observation: Given an entanglement witness W,

$$\mathrm{tr}(\rho W) < 0 \implies \rho \text{ entangled.}$$

Notes:

- Key point: Need some way to prove that $\mathrm{tr}(W\rho) \geq 0 \quad \forall \rho \in \mathcal{S}$!
- Any given witness will only detect some entangled states.
- $\mathcal{S}$ is a convex set $\implies$ $\mathcal{S}$ is fully specified by all its tangent planes $\implies$ there exists a witness for any entangled states.

- Witnesses are <u>linear operators</u> $\Rightarrow$ they can be experimentally measured and can (and are) thus being used to identify entanglement in <u>experiments</u>.

<u>Example:</u>

$$W = \mathbb{F} := \sum_{i,j=1}^{d} |i,j\rangle\langle j,i| \quad \text{(the "flip" or swap)}$$

Is it a witness?

Let $\rho_{sep} = \sum p_i \rho_i^A \otimes \rho_i^B$:

$$tr(W \rho_{sep}) = \sum p_i \, tr\left(\mathbb{F} \left(\rho_i^A \otimes \rho_i^B\right)\right) = \dots$$

Use:

(1) $\underline{tr[\mathbb{F} \cdot (A \otimes B)]} = \sum_{ij} tr\left[|ij\rangle\langle ji| \, A \otimes B\right]$

$$= \sum_{ij} \langle j,i| (A \otimes B) |i,j\rangle = \sum_{ij} \langle j|A|i\rangle\langle i|B|j\rangle$$

$$= \underline{tr(AB)} \quad \text{(the "magic formula")},$$

(2) $P, Q \geq 0 \implies \mathrm{tr}(PQ) \geq 0$

<u>Proof</u>: $P = \sum p_i |\phi_i\rangle\langle\phi_i|$, $p_i \geq 0$.

$$\mathrm{tr}(PQ) = \sum_i \mathrm{tr}(p_i |\phi_i\rangle\langle\phi_i| Q)$$

$$= \sum_i p_i \underbrace{\langle\phi_i|Q|\phi_i\rangle}_{\geq 0} \geq 0.$$

$$\ldots = \sum p_i \underbrace{\mathrm{tr}(\rho_i^A \rho_i^B)}_{\geq 0} \geq 0.$$

$\implies W$ is entanglement witness!

Which ent. states does W detect? — Those with dominant anti-symmetric component!

E.g. $d=2$, $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$:

$$\mathbb{F}|\psi^-\rangle = -|\psi^-\rangle \implies \langle\psi^-|\mathbb{F}|\psi^-\rangle = -1 < 0 \,!$$

while for e.g. $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$,

$$\mathbb{F}|\phi^+\rangle = |\phi^+\rangle \longrightarrow \langle\phi^+|\mathbb{F}|\phi^+\rangle = +1 > 0 \text{ ↯}$$

What about <u>mixed states</u>?

E.g. for $d=2$:

$$\rho = \lambda |\psi^-\rangle\langle\psi^-| + (1-\lambda)\frac{I}{4} \quad ; \quad \lambda \in [-\tfrac{1}{3}, 1]$$

"<u>Werner state</u>"

$$\mathrm{tr}(\mathbb{F}\rho) = \lambda \langle\psi^-|\mathbb{F}|\psi^-\rangle + \frac{(1-\lambda)}{4}\underbrace{\mathrm{tr}(I\cdot\mathbb{F})}_{=d=2}$$

$$= -\lambda + \frac{1-\lambda}{2} = \frac{1}{2}(1-3\lambda)$$

$\rightarrow$ for $\lambda > 1/3$, ρ is entangled

<u>Definition</u>: A witness W is called <u>optimal</u> if there exist ρ separable such that $\mathrm{tr}[W\rho] = 0$.

(i.e., it touches the convex set & cannot be moved closer; otherwise, we could move it in parallel and get a strictly better witness.)

Is $W = \mathbb{F}$ optimal?

Yes, e.g. $\rho = |0\rangle\langle 0| \otimes |1\rangle\langle 1|$

$\Rightarrow \mathrm{tr}(\rho \mathbb{F}) = 0.$

Other witnesses: E.g. $W = I - d|\Omega\rangle\langle\Omega|$,

$$|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i,i\rangle \longrightarrow \text{Homework!}$$

c) Positive maps and the PPT criterion

Reminder: A superoperator $\Lambda : B(\mathcal{H}) \longrightarrow B(\mathcal{H})$ is called positive if $\rho \geq 0 \Rightarrow \Lambda(\rho) \geq 0$.

Usually – i.e. for physical maps – we want Λ in addition to be completely positive, i.e.

$$\rho_{AB} \geq 0 \Rightarrow (\Lambda_A \otimes I_B)(\rho_{AB}) \geq 0.$$

But now we will be interested in positive but not completely positive maps!

Why?

Consider $\rho_{sep} = \sum p_i \rho_i^A \otimes \rho_i^B$. Then,

$$(\Lambda \otimes I)(\rho_{sep}) = \sum p_i \underbrace{\Lambda(\rho_i^A)}_{=:\tilde{\rho}_i^A \geq 0 \ (\Lambda \text{ positive!})} \otimes \rho_i^B$$

$$= \sum \underbrace{p_i}_{\geq 0} \underbrace{\tilde{\rho}_i^A}_{\geq 0} \otimes \underbrace{\rho_i^B}_{\geq 0} \geq 0$$

and thus:

Theorem: Let $\Lambda : B(\mathcal{H}) \to B(\mathcal{H})$ be a positive map.

Then, $(\Lambda \otimes I)(\rho_{AB}) \ngeq 0 \implies \rho_{AB}$ entangled.

not positive semi-definite, i.e. has negative eigenvalues.

Most important example:

$$\Lambda(\rho) = \rho^T \quad (\text{the } \underline{\text{transpose map}})$$

$$(\Lambda \otimes I)(\rho) =: \rho^{T_A}$$ "partial transpose"

(cf. also section II.5)

E.g.: $|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i,i\rangle$

$$\Rightarrow (|\Omega\rangle\langle\Omega|)^{T_A} = \frac{1}{d}\left(\sum |i,i\rangle\langle j,j|\right)^{T_A}$$

$$= \frac{1}{d} \underbrace{\sum |j,i\rangle\langle i,j|}_{= \mathbb{F}!}$$

Not positive – antisymm. states have neg. eigenvalue,

i.e.: $|A\rangle = \frac{1}{\sqrt{2}}\left(|ij\rangle - |ji\rangle\right)$, $i \neq j$:

$$\langle A| (|\Omega\rangle\langle\Omega|)^{T_A} |A\rangle = \frac{1}{d}(-1) = -\frac{1}{d}.$$

Mixed states:

E.g. $\rho = \lambda |\Omega\rangle\langle\Omega| + (1-\lambda)\frac{I}{d^2}$, $\frac{-1}{d^2-1} \leq \lambda \leq 1$:

"isotropic state"

For instance for $d=2$:

$$\rho = \lambda \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} + (1-\lambda) \begin{pmatrix} \frac{1}{4} & & & \\ & \frac{1}{4} & & \\ & & \frac{1}{4} & \\ & & & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1+\lambda}{4} & & & \frac{\lambda}{2} \\ & \frac{1-\lambda}{4} & & \\ & & \frac{1-\lambda}{4} & \\ \frac{\lambda}{2} & & & \frac{1+\lambda}{4} \end{pmatrix}$$

$$\Longrightarrow \rho^{T_A} = \begin{pmatrix} \frac{1+\lambda}{4} & & & \\ & \boxed{\begin{matrix} \frac{1-\lambda}{4} & \frac{\lambda}{2} \\ \frac{\lambda}{2} & \frac{1-\lambda}{4} \end{matrix}} & \\ & & & \frac{1+\lambda}{4} \end{pmatrix}$$

positive iff $\left(\frac{1-\lambda}{4}\right)^2 \geq \left(\frac{\lambda}{2}\right)^2$ & $\frac{1\pm\lambda}{4} \geq 0$

$$\Longleftrightarrow -1 \leq \lambda \leq 1/3$$

$$\Longrightarrow \rho^{T_A} \geq 0 \text{ iff } \lambda \in \left[\frac{-1}{d^2-1} ; \frac{1}{3}\right]$$

$\Longrightarrow$ For $\lambda \in \left(\frac{1}{3} ; 1\right]$, ρ is entangled!

Corollary (PPT criteria):

$\rho^{T_A} \ngeq 0 \Longrightarrow \rho$ entangled.

This is called the PPT (positive partial transpose) criteria, or also NPT criteria. (not positive...).

Note: PPT criterion – and generally positive maps – are invariant under local unitaries on B $\Longrightarrow$ PPT detects *all* maximally ent. states (and in fact all states $|\chi\rangle$ w/ full Schmidt rank, as $|\chi\rangle = (I_A \otimes M_B)|\Omega\rangle$, M_B invertible).

That is: Positive maps are *stronger* than witnesses (will formalize this later) – but they cannot be measured.

In fact:

Theorem: The PPT criterion detects *all* entangled states in $d_A \times d_B = 2\times 2$ and 3×2 dimensions,

$$\rho \text{ entangled} \iff \rho^{T_A} \not\geq 0$$

(Not proven here.)

Counterexamples exist in $d_A \times d_B = 3\times 3$ and 2×4, i.e. states ρ with $\rho^{T_A} \geq 0$ which are entangled.

("PPT bound entangled states", cf. later.)

Other example of a positive but not CP map:

$$\Lambda(\rho) = \mathrm{tr}(\rho) I - \rho$$

$$(\Lambda \otimes I)(\rho_{AB}) = (I \otimes \mathrm{tr}_A \rho_{AB}) - \rho_{AB}.$$

$$I \otimes \mathrm{tr}_A \rho_{AB} - \rho_{AB} \not\geq 0 \Rightarrow \rho \text{ entangled.}$$

The "reduction criterion" for entanglement:

$$\underline{I \otimes \mathrm{tr}_A \rho \geq \rho} \qquad (\to \text{Homework}).$$

d) Relation between witnesses and positive maps

For each witness W, there is a positive map which detects all entangled states the witness detects — and in fact more. More precisely:

Theorem: There is an isomorphism between witnesses W on $\mathbb{C}^d \times \mathbb{C}^d$ and positive maps $\Lambda : B(\mathbb{C}^d) \to B(\mathbb{C}^d)$.
Moreover, if $tr(W\rho) < 0 \Rightarrow (\Lambda \otimes I)(\rho) \not\geq 0$.

Proof:

Reminder (Choi-Jam. isomorphism):

CP map $\mathcal{E}$ on A $\qquad$ state σ on AB $(d_A = d_B)$

$$\mathcal{E} \longmapsto \sigma = (\mathcal{E} \otimes I)(|\Omega\rangle\langle\Omega|)$$

$$\mathcal{E}(\rho) = d\, tr_B\left(\sigma (I \otimes \rho^T)\right) \longleftarrow \sigma$$

Isomorphism $\mathcal{E} \rightleftarrows \sigma$ applies also outside of CP maps and $\sigma \geq 0$.

Proof idea: Interpret W^T as the "Choi state" of a map Λ. (Since $W \not\geq 0$ — otherwise $tr(W\rho) \geq 0\ \forall \rho$! — Λ is not CP.)

i.e.: $\Lambda(X) := d \, tr_B \left(W^T (I \otimes X^T) \right)$

$$= d \, tr_B \left(W (I \otimes X) \right)^T .$$

Need to show:

i) W witness $\Rightarrow$ $\Lambda(X)$ positive map.

ii) $\Lambda(X)$ positive map $\Rightarrow$ W witness

iii) $tr(W\rho) < 0 \Rightarrow (\Lambda \otimes I)(\rho) \ngeq 0$.

Proof of i): Let $\rho \geq 0$. Then,

$$\langle \phi | \Lambda(\rho)^T | \phi \rangle = d \langle \phi | \, tr_B \left(W (I \otimes \rho) \right) | \phi \rangle$$

$$= d \, tr \Big[W \big(\underbrace{|\phi\rangle\langle\phi| \otimes \rho}_{\text{sep. state!}} \big) \Big] \geq 0$$

W witness!

$\Rightarrow$ Λ is a positive map!

For ii) and iii), use

$$(*) \quad \begin{aligned} tr\left[W (A \otimes B) \right] &= tr_A \left[tr_B \left(W (I \otimes B) \right) \cdot A \right] \\ &= \frac{1}{d} tr_A \left[\Lambda(B)^T A \right] = \frac{1}{d} \sum_{ij} \left[\Lambda(B)^T \right]_{ij} A_{ji} \\ &= \langle \Omega | A \otimes \Lambda(B) | \Omega \rangle \end{aligned}$$

Proof of ii) Let $\rho_i^A, \rho_i^B \geq 0$; $p_i \geq 0$. Then,

$$tr\left[W\left(\sum p_i \rho_i^A \otimes \rho_i^B\right)\right] = \sum p_i \, tr\left[W(\rho_i^A \otimes \rho_i^B)\right]$$

$$\overset{(*)}{=} \sum p_i \langle \Omega | \underbrace{\rho_i^A}_{\geq 0} \otimes \underbrace{\Lambda(\rho_i^B)}_{\geq 0, \text{ as } \Lambda \text{ positive.}} | \Omega \rangle \geq 0$$

$\Rightarrow$ W is witness.

Proof of iii): $(*)$ together with linearity $\Rightarrow$

$$\Rightarrow \quad tr(W\rho) = \langle \Omega | (I \otimes \Lambda)(\rho) | \Omega \rangle$$

which implies

$$tr(W\rho) < 0 \Rightarrow (I \otimes \Lambda)(\rho) \not\geq 0$$

$\Rightarrow$ Λ detects all states which W detects. $\blacksquare$

Conversely: Checking the witness W amounts to using the positive map Λ, but checking for negative eigenvalues only along $|\Omega\rangle$ $\Rightarrow$ weaker criterion!

Corollary: A state is separable if and only if

$(\Lambda \otimes I)(\rho) \geq 0 \quad \forall$ positive maps Λ

(since sep. states $\leftrightarrow$ witnesses $\leftrightarrow$ positive maps).

Example: $W = \mathbb{F}$.

$$\Lambda(X) = d \, \mathrm{tr}_B \left(\mathbb{F} \left(I \otimes X^T \right) \right) = d \cdot X^T \cdot I = d X^T$$

"magic formula" for partial trace!

$\Longrightarrow W = \mathbb{F}$ corresponds to PPT criterion!

(But: PPT detect all Bell states, $\mathbb{F}$ only the antisym. state)

e) Quantification of mixed state entanglement

How can we quantify entanglement of mixed states?

i) Entanglement needed to create the state, e.g. minimal amount of $E(|\psi\rangle) = S(\mathrm{tr}_B |\psi\rangle\langle\psi|)$ needed:

$$E_F(\rho) := \min_{\substack{\{p_i, |\psi_i\rangle\} \\ \text{s.th. } \sum p_i |\psi_i\rangle\langle\psi_i| = \rho}} \sum p_i E(|\psi_i\rangle)$$

"entanglement of formation"

or an asymptotic version (cost per copy)

$$E_C(\rho) := \lim_{N\to\infty} \frac{1}{N} E_F(\rho^{\otimes N})$$

"entanglement cost"

Already E_F very hard to compute – need to minimize concave function $S(\rho)$ over convex set

of decompositions $\rho = \sum p_i \rho_i$ – i.e. ~~[illegible]~~
obtained at boundary (hard!)

Also, $E_F(\rho \otimes \sigma) \neq E_F(\rho) + E_F(\sigma)$
(additivity of E_F – counterex. exist (Hastings))

<u>But:</u> <u>Analytic formula</u> for <u>$d_A \times d_B = 2\times 2$</u> for E_F
exists (Wootters), based on so-called "<u>concurrence</u>".

ii) <u>Extractable entanglement:</u>

"<u>Distillable entanglement</u>" $E_D(\rho)$

$E_D(\rho)$ = max. asympt. rate $R = \frac{M}{N}$ achievable
for LOCC-protocol $\mathcal{E}_N$: $\| \mathcal{E}_N(\rho^{\otimes N}) - |\phi^+\rangle\langle\phi^+|^{M} \| \to 0$
as $N, M \to \infty$.

suitable dist. meas, typ. trace norm.

<u>Even harder</u> to compute:

Asymptotic ($N, M \to \infty$) and any # of LOCC rounds.

(Versions w/ restricted LOCC rounds exist, e.g. one-way dist. entanglement, ...)

<u>Observation:</u> $E_F(\rho) \geq E_C(\rho) \geq E_D(\rho)$

Generally, $E_C(\rho) \not\geq E_D(\rho)$: for most states, process is <u>not reversible</u>.

$\Longrightarrow$ no unique measure as for pure states!

<u>Example:</u> ρ PPT, i.e. $\rho^{T_A} \geq 0$.

LOCC preserves PPT $\Longrightarrow$ PPT states are <u>undistillable</u>, $E_D(\rho) = 0$.

But there exist PPT states with $E_C(\rho) > 0$:

"PPT bound entangled states"

(<u>Note:</u> The converse problem –

does $\rho^{T_A} \not\geq 0 \Longrightarrow E_D(\rho) > 0$ hold?

– is a big open problem – the existence of NPT bound entangled states.)

Problem: These might be natural ent. measures, but they are essentially impossible to compute.

→ Computable ent. measure desirable!

Wishlist for a good ent. measure:

- LOCC-monotone: Cannot be increased by LOCC (probably most relevant!)
- $E(\rho)=0$ for sep. states ρ (and only there?)
- additive: $E(\rho\otimes\sigma) = E(\rho)+E(\sigma)$
- continuous: $\rho \approx \sigma \Rightarrow E(\rho)\approx E(\sigma)$

  typ. trace norm
- $E_D \leq E \leq E_C$.
- coincides with $E(|\psi\rangle) = S(\mathrm{tr}_B |\psi\rangle\langle\psi|)$ on pure states.
- computable?

(Almost) impossible to get all - LOCC monotonicity is probably the most relevant one.

Negativity – a computable entanglement measure

Found previously:

ρ^{T_A} has neg. eigenvalues $\Rightarrow$ ρ entangled.

Use negative eigenvalues to <u>measure</u> entanglement:

"**Negativity**"
$$\mathcal{N}(\rho) = \frac{1}{2}\Big(\underbrace{\sum_i \big|\overset{\text{eigenvalues}}{\lambda_i}(\rho^{T_A})\big|}_{=:\,\|\rho^{T_A}\|_1 \;:\text{ trace norm}} - 1 \Big)$$

$$= \frac{1}{2}\left(\|\rho^{T_A}\|_1 - 1 \right)$$

$$= - \sum_{\text{neg. eigenvalues}} \lambda_i(\rho^{T_A})$$

holds since $\sum \lambda_i(\rho^{T_A}) = tr(\rho^{T_A})$
$= tr(\rho) = 1$

or "**Log-negativity**"

$$E_N(\rho) = \log_2 \|\rho^{T_A}\|_1$$

Properties:

Negativity $\mathcal{N}$:

- LOCC - monotone
- not additive
- $\mathcal{N}(\rho)=0$ for ρ separable,
  but also $\mathcal{N}(\rho)$ on PPT ent. states!
- $\neq E(|\psi\rangle)$ for pure states
- continuous

Log-negativity E_N:

- not an LOCC monotone (b)
- additive
- $E_N(\rho)=0$ for ρ sep, but also on PPT ent. st.
- $\neq E(|\psi\rangle)$ for pure states
- continuous