2. Oracle-based algorithms

a) The Deutsch algorithm

Consider $f: \{0,1\} \longrightarrow \{0,1\}$

Let f be "very hard to compute" – e.g. long circuit

Want to know: Is $f(0) = f(1)$?

(e.g.: will a specific chess move affect result?)

How often do we have to run the circuit for f (= "evaluate f")? — We think of f as a "black box" or "oracle": How many oracle queries are needed?

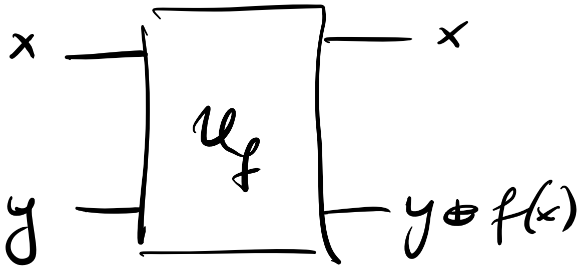
Classically, we clearly need 2 queries:

compute $f(0)$ and $f(1)$.

Can quantum physics help?

Consider reversible implementation of f:

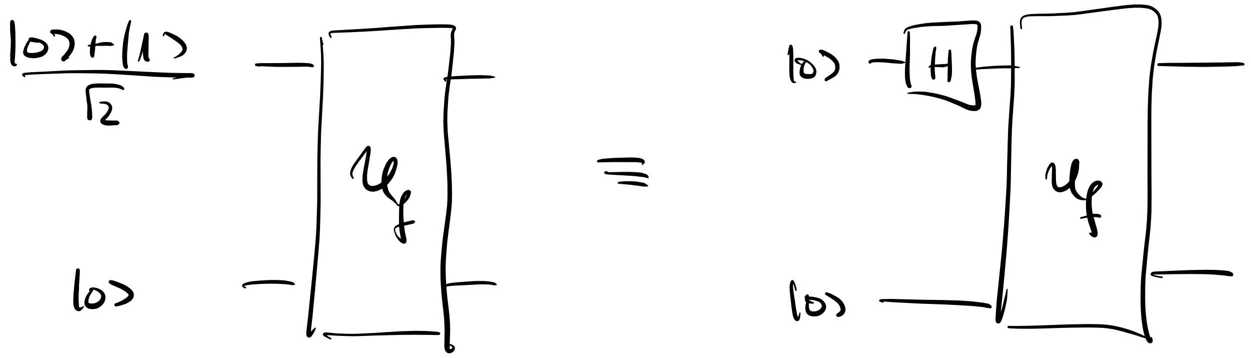
$$f^R: (x, y) \longmapsto (x, y \oplus f(x))$$



$$|x\rangle|y\rangle \longmapsto |x\rangle|y \oplus f(x)\rangle$$

Try to use superpositions as inputs?

First attempt:



$$\frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes |0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|0\rangle\right) \xmapsto{U_f} \frac{1}{\sqrt{2}}\left(|0\rangle|f(0)\rangle + |1\rangle|f(1)\rangle\right)$$

→ Have evaluated f on both inputs!

But how can we extract the relevant information

(i.e. do a measurement)?

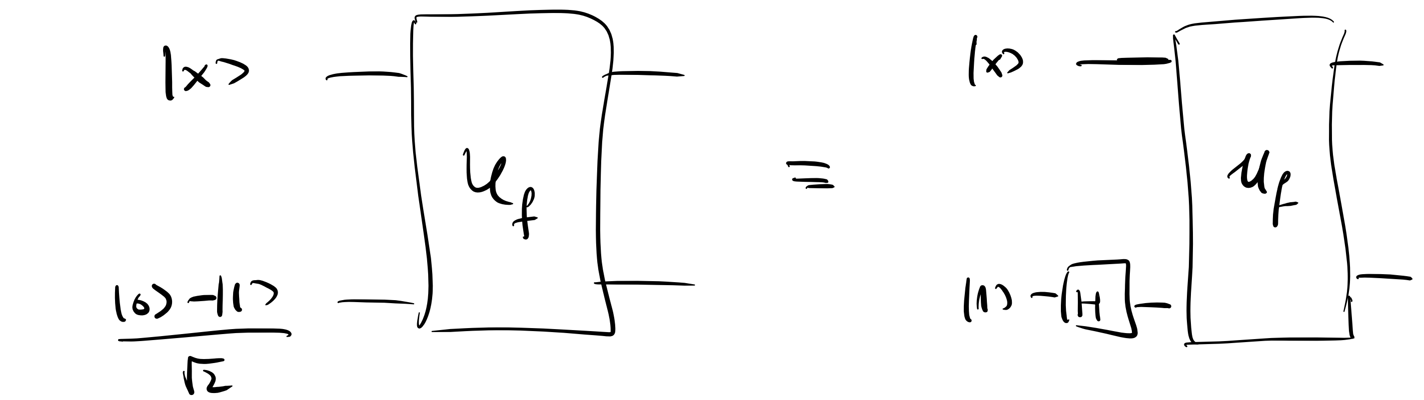
- Meas. in comp. basis: collapse superpos. to one case!
- Generally: $f(0) \neq f(1)$: outputs $\frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|1\rangle\right)$,

  $\frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle + |1\rangle|0\rangle\right)$,

$\underline{f(0) = f(1)}$: outputs $|+\rangle|0\rangle$,
$|+\rangle|1\rangle$.

$\Longrightarrow$ not orthogonal, i.e. not (determ.) distinguishable!

<u>Second attempt:</u>



$$|x\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) \xmapsto{\mathcal{U}_f} |x\rangle\left(\frac{|f(x)\rangle - |1\oplus f(x)\rangle}{\sqrt{2}}\right) =$$

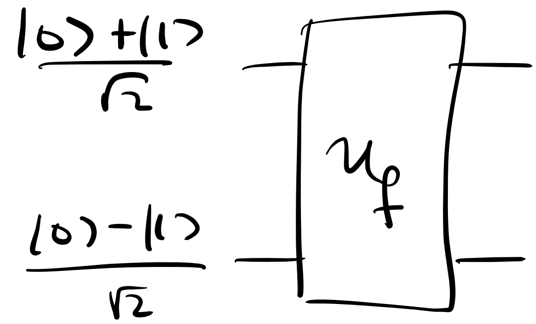
$$= \begin{cases} f(x)=0: & |x\rangle \dfrac{|0\rangle - |1\rangle}{\sqrt{2}} \\[2ex] f(x)=1: & |x\rangle \dfrac{|1\rangle - |0\rangle}{\sqrt{2}} \end{cases}$$

$$= |x\rangle\left[(-1)^{f(x)}\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right]$$

$$= (-1)^{f(x)} |x\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

Not useful by itself: $f(x)$ only encoded in <u>global phase</u> for each <u>classical input</u> <u>$|x\rangle$</u>.

<u>Combine attempts:</u>



:

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \left(|0\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}} + |1\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

$$= \frac{1}{\sqrt{2}} \left((-1)^{f(0)} |0\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}} + (-1)^{f(1)} |1\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

$$= \frac{(-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

<u>Observations:</u>

→ No entanglement created (!)

→ 2nd qubit – the one where U_f outputs the function value – is unchanged (!!)

→ 1st qubit gets a phase $(-1)^{f(x)}$

("<u>phase kick-back technique</u>")

<u>State of 1st qubit:</u>

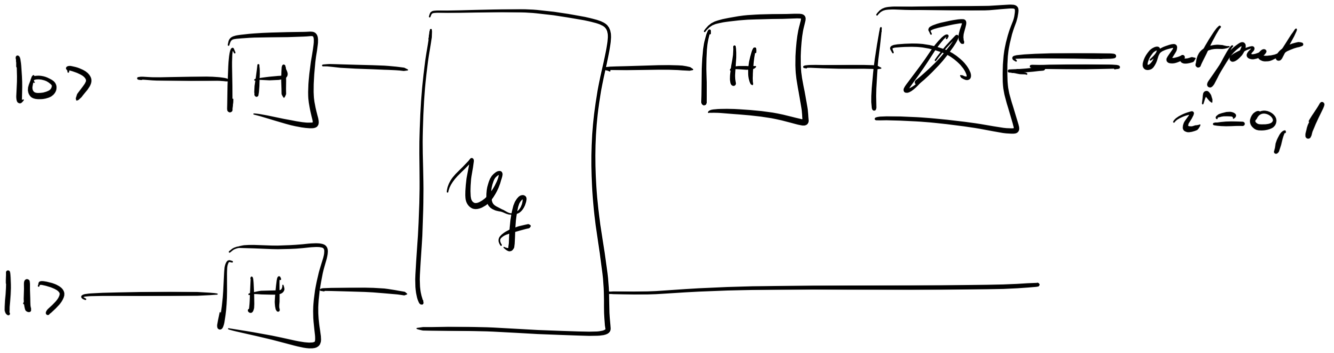
$$f(0) = f(1) \Longleftrightarrow \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$f(0) \neq f(1) \Longleftrightarrow \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

(up to irrelevant global phase)

<u>Orthogonal states!</u> $\Longrightarrow$ measurement of 1st qubit in basis $\{|+\rangle, |-\rangle\}$ (or apply $-\boxed{H}-$ & measure in $\{|0\rangle, |1\rangle\}$) allows to decide if $f(0) \overset{?}{=} f(1)$!

<u>Deutsch algorithm:</u>



output $i=0: \Rightarrow f(0)=f(1)$

$i=1: \Rightarrow f(0) \neq f(1)$

One application of U_f has been sufficient!

$\Rightarrow$ Speed-up compared to class. algorithm

(1 vs. 2 oracle queries).

Interesting to note: 2nd qubit never needs to be measured — and it contains no information.

Two main insights:

- Use input $\sum |x\rangle$ to evaluate f on all inputs simultaneously.
- This parallelism alone is not enough — need a smart way to read out the relevant information.

However, a constant speed-up is not that impressive — in particular, it is highly architecture-dependent!

Thus:

b) The Deutsch-Jozsa algorithm

Consider $f: \{0,1\}^n \to \{0,1\}$ with promise (i.e., a condition we know is met by f) that

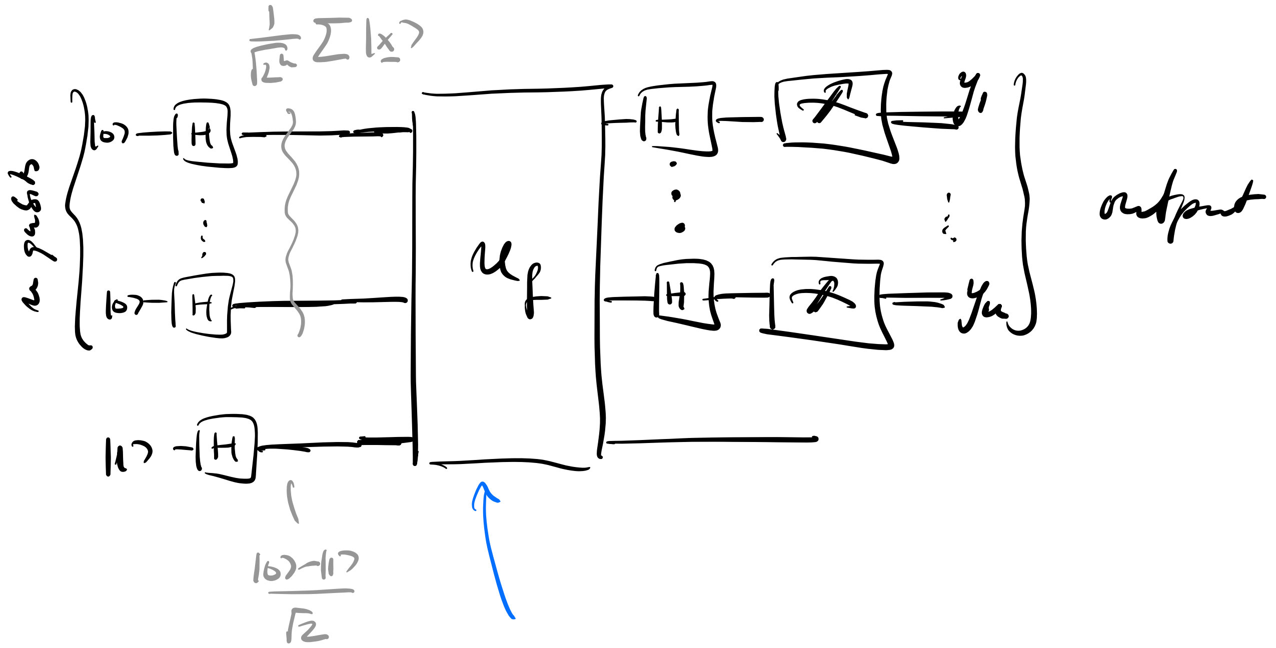
either $f(\underline{x}) = c \quad \forall \underline{x}$ ("f constant")

or $|\{\underline{x} \mid f(\underline{x}) = 0\}| = |\{\underline{x} \mid f(\underline{x}) = 1\}|$ ("f balanced")

Want to know: Is f constant or balanced?

How many queries needed?

Use same idea: Input $\sum |x\rangle$ and $\frac{|0\rangle - |1\rangle}{\sqrt{2}}$:



$U_f: |\underline{x}\rangle|y\rangle \longmapsto |\underline{x}\rangle|y \oplus f(x)\rangle$

Before analyzing circuit: What is action of $H^{\otimes 4}$?

$$H: \ |x\rangle \longmapsto \frac{1}{\sqrt{2}} \sum_{y=0,1} (-1)^{x \cdot y} |y\rangle$$

$$H^{\otimes n}: \ |x_1, \ldots, x_n\rangle \longmapsto \frac{1}{\sqrt{2^n}} \sum_{\underline{y}} (-1)^{x_1 y_1} \cdot \ldots \cdot (-1)^{x_n y_n} |y_1, \ldots, y_n\rangle$$

or:

$$|\underline{x}\rangle \longmapsto \frac{1}{\sqrt{2^n}} \sum (-1)^{\underline{x} \cdot \underline{y}} |\underline{y}\rangle$$

where $\underline{x} \cdot \underline{y} := x_1 y_1 \oplus x_2 y_2 \oplus \ldots \oplus x_n y_n$

("scalar product" mod 2).

is NOT a scalar product!

Analysis of circuit:

we omit normalization!

$$|\underline{0}\rangle|1\rangle \xrightarrow{H^{\otimes n} \otimes H} \left(\sum_{\underline{x}} |\underline{x}\rangle \right) \left(|0\rangle - |1\rangle \right)$$

phase kick-back

$$\xrightarrow{U_f} \left(\sum_{\underline{x}} (-1)^{f(\underline{x})} |\underline{x}\rangle \right) \left(|0\rangle - |1\rangle \right)$$

$$\xrightarrow{H^{\otimes n} \otimes I} \left(\sum_{\underline{y}} \underbrace{\sum_{\underline{x}} (-1)^{f(\underline{x}) + \underline{x} \cdot \underline{y}}}_{=: a_{\underline{y}}} |\underline{y}\rangle \right) \left(|0\rangle - |1\rangle \right)$$

$p_{\underline{y}} := |a_{\underline{y}}|^2$ is the probability to measure $\underline{y} = (y_1, \dots, y_n)$.

f constant: $f(\underline{x}) = c$

$$a_{\underline{y}} = (-1)^c \underbrace{\sum_{\underline{x}} (-1)^{\underline{x} \cdot \underline{y}}}_{\propto \delta_{\underline{y}, \underline{0}}} = (-1)^c \delta_{\underline{y}, \underline{0}}$$

f balanced:

For $\underline{y} = \underline{0}$: $$a_{\underline{0}} = \sum_{\underline{x}} (-1)^{f(\underline{x}) + \underline{x} \cdot \underline{0}}$$

$$= \sum_{x} (-1)^{f(\underline{x})} = 0$$

f balanced!

Thus:

Output $\underline{y} = \underline{0}$ $\Longrightarrow$ f constant

Output $\underline{y} \neq \underline{0}$ $\Longrightarrow$ f balanced

$\Longrightarrow$ We can unambiguously distinguish the 2 cases with one query to the oracle for f!

What is the speed-up vs. classical methods?

Quantum: 1 use of f.

Classical: Worst case, we have to determine

$2^{n-1}+1$ values of f to be sure!

$\Rightarrow$ exponential vs. constant!

But: If we are ok to get right answer with very high probability $p = 1 - P_{error}$, then for k queries to f,

$$P_{error} \approx 2 \cdot \underbrace{\left(\frac{1}{2}\right)^k}$$

$\approx$ prob. to get $k\times$ same outcome for balanced f, if $k \ll 2^n$.

i.e.: $k \sim \log(1/P_{error})$.

Randomized classical: Much smaller speed-up vs. randomized classical algorithm (even for exp. small error, $k \sim n$ oracle calls are sufficient.)

c) Simon's algorithm

... will give us a true exponential speedup (also rel. to randomized class. algorithms) in terms of oracle queries!

Oracle: $f : \{0,1\}^n \longrightarrow \{0,1\}^n$

with promise:

$\exists\, \underline{a} \neq \underline{0}$ s.th. $f(\underline{x}) = f(\underline{y})$ exactly if $\underline{y} = \underline{x} \oplus \underline{a}$.

("hidden periodicity")

Task: Find $\underline{a}$ by querying f.

Classical: Need to query $f(\underline{x}_i)$ until pair $\underline{x}_i, \underline{x}_j$ with $f(\underline{x}_i) = f(\underline{x}_j)$ is found.

Roughly: k queries $x_1, \dots, x_k \longrightarrow \sim k^2$ pairs,

for each pair: prob $(f(\underline{x}_i) = f(\underline{x}_j)) \approx 2^{-n}$

$\Longrightarrow P_{success} \sim k^2\, 2^{-n}$

$\Rightarrow$ need $k \sim 2^{cn}$ queries!

Quantum algorithm (Simon's algorithm):

i) Start with $\frac{1}{\sqrt{2^n}} \sum_{\underline{x}} |\underline{x}\rangle = H^{\otimes n} |\underline{0}\rangle$

ii) Apply $U_f : |\underline{x}\rangle|y\rangle \longmapsto |\underline{x}\rangle|y \oplus f(\underline{x})\rangle$

$$U_f : \left(\frac{1}{\sqrt{2^n}} \sum_{\underline{x}} |\underline{x}\rangle_A \right) |\underline{0}\rangle_B \longmapsto \frac{1}{\sqrt{2^n}} \sum_{\underline{x}} |\underline{x}\rangle_A |f(\underline{x})\rangle_B$$

iii) Measure B. $\Rightarrow$ Collapse onto random $f(\underline{x}_0)$ (and thus random $\underline{x}_0$).

$\Rightarrow$ Register A collapses onto

$$\frac{1}{N} \sum_{\underline{x} : f(\underline{x}) = f(\underline{x}_0)} |x\rangle = \frac{1}{\sqrt{2}} \left(|\underline{x}_0\rangle + |\underline{x}_0 \oplus \underline{a}\rangle \right)$$

— How can we extract a? —

(Meas. in comp. basis $\rightarrow$ collapse on rand. x_0: useless.)

iv) Apply $H^{\otimes n}$ again:

$$\boxed{H^{\otimes n} |\underline{x}\rangle \propto \sum_{y} (-1)^{\underline{x} \cdot \underline{y}} |\underline{y}\rangle}$$

$$H^{\otimes n} \left(\frac{1}{\sqrt{2}} \left(|\underline{x}_0\rangle \oplus |\underline{x}_0 \oplus a\rangle \right) \right)$$

$$= \frac{1}{\sqrt{2^{n+1}}} \sum_{y} \underbrace{\left[(-1)^{\underline{x}_0 \cdot y} + (-1)^{(\underline{x}_0 + \underline{a}) \cdot y} \right]}_{} |y\rangle$$

$\rightarrow \underline{a} \cdot y = 0 \Rightarrow \quad = 2 \cdot (-1)^{\underline{x}_0 \cdot y}$

$\underline{a} \cdot y = 1 \Rightarrow \quad = 0$

$$= \frac{1}{\sqrt{2^{n-1}}} \sum_{y : \underline{a} \cdot y = 0} (-1)^{\underline{x}_0 \cdot y} |y\rangle$$

v) Measure in comp. basis:

$\Rightarrow$ obtain random y s.th. $\underline{a} \cdot y = 0$.

$(n-1)$ lin. indep. vectors y_i (over $\mathbb{Z}_2$) s.th. $\underline{a} \cdot y_i = 0$ allow to determine $\underline{a}$ (solve lin. eq. – e.g. Gaussian elimination).

Space of lin. dep. vectors of k vectors grows as 2^k

$\Rightarrow O(1)$ chance to find randomly a lin. indep. vector

$\Rightarrow O(n)$ random y are enough

$\Rightarrow$ $O(n)$ oracle queries are enough (on average)

Classical: 2^{cn} queries
Quantum: $c' n$ queries

$\Big\}$ exponential speed-up !

(in terms of oracle queries)

Notes: • We don't have to measure B — we never use the outcome! (But: Derivation easier this way!)

• $H^{\otimes n} \mathrel{\hat{=}}$ (discrete) Fourier transform over $\mathbb{Z}_2^{\times n}$

$\longrightarrow$ period finding via Fourier transform