V. Quantum Error Correction

1. Introduction

a) Setting & Problem

- Coupling to environment induces errors (i.e., uncontrolled behavior).
- Classical computers: Information stored in "macroscopic" properties $\longrightarrow$ errors unlikely.
- Quantum computers:
  - need qubits = "single" quantum systems, and must store general superposition, not just $|0\rangle$ and $|1\rangle$

    $\longrightarrow$ fragile!
  - should be well isolated to protect qubits, but also need coupling to "environment" (experimental apparatus) to control the computation (gates, measurements).

Q: Can we protect quantum information from noise?

Classical error correction:

copy information, e.g. encode 1 bit in 3 bits:

$$0 \longmapsto \hat{0} := 000$$
$$1 \longmapsto \hat{1} := 111$$

"encoding"

Error model: Bit flip w/ some (small) probability p (independently on all bits):

$\Rightarrow$ typically 0 or 1 bits flipped.

Error correction ("decoding") by majority vote:

$$000, 001, 010, 100 \longmapsto 000$$
$$111, 110, 101, 011 \longmapsto 111$$

Probability for a "logical error" (i.e. on encoded bit):

$$p_{error} = \text{prob}(\geq 2 \text{ flips}) = p^3 + 3p^2(1-p)$$
$$= 3p^2 - 2p^3 < p \quad \text{for } p < 1/2.$$

error quadratically suppressed!

$\Rightarrow$ effective error probability decreased.

Can be improved by:

- using more bits: $0 \mapsto 00\cdots0$, $1 \mapsto 11\cdots1$
- nesting ("concatenating") codes
- using smarter codes (i.e. encode several bits at once)

Quantum Error Correction:

Several potential problems when trying to generalize classical error correction codes:

- cannot copy qubits
- even if we could: what would be the "majority vote"?
- different types of errors exist,

  e.g. X (bit flip)

  or Z ("phase flip")
- errors can be continuous: there is an infinity of errors!
- measuring qubits destroys quantum information!

6) The 3-qubit bit flip code

Copy qubits in computational basis:

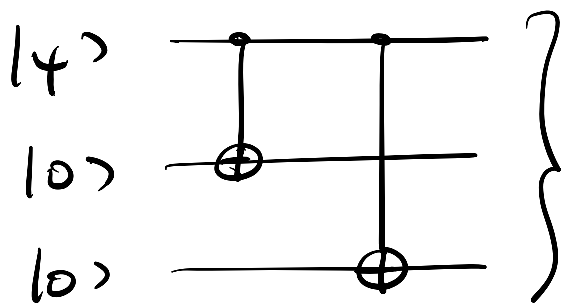
$$|0\rangle \longmapsto |\hat{0}\rangle = |000\rangle$$

$$|1\rangle \longmapsto |\hat{1}\rangle = |111\rangle$$

i.e., the encoding is a linear map

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \xmapsto{\text{encoding}} \alpha|000\rangle + \beta|111\rangle$$

Possible encoding circuit:



$$|\hat{\psi}\rangle = \alpha|000\rangle + \beta|111\rangle$$

Now consider bit flip error on qubit i:

$$|\hat{\psi}\rangle \xmapsto{\text{error}} X_i|\hat{\psi}\rangle$$

Can we correct for one bit flip error on an unknown qubit i?

Problem: Measuring the qubits in comp. basis reveals i, but also destroys superposition!

$\Longrightarrow$ Need a measurement which only returns information about position i of error – indep. of encoded state $|\psi\rangle$!

Define "syndrome measurement" with outcomes 0, 1, 2, 3, and projectors:

0 = "no flip": $P_0 = |000\rangle\langle 000| + |111\rangle\langle 111|$

1 = "1st qubit flipped": $P_1 = |100\rangle\langle 100| + |011\rangle\langle 011|$

2 = "2nd qubit flipped": $P_2 = |010\rangle\langle 010| + |101\rangle\langle 101|$

3 = "3rd qubit flipped": $P_3 = |001\rangle\langle 001| + |110\rangle\langle 110|$

(This defines a complete measurement, as $\sum P_i = I$)

The outcome is called the "error syndrome".

Measurement of $\{P_\alpha\}$ reveals only 2 bits of info.

$\Longrightarrow$ one qubit of information untouched!

By direct inspection: The information obtained is the location of the bit flip, e.g.

$$\alpha|000\rangle + \beta|111\rangle \xmapsto[\text{on qubit } 2]{\text{bit flip}} \alpha|010\rangle + \beta|101\rangle$$

$\Rightarrow$ measurement always returns P_2, with post-measurement state

$$\alpha|010\rangle + \beta|101\rangle \xmapsto[\text{flip qubit } 2]{\text{recovery:}} \alpha|000\rangle + \beta|111\rangle \text{ !}$$

$\Rightarrow$ Bit flip corrected!

Works for any single bit flip in unknown location and no flip, and for all states $|\psi\rangle$

$\Rightarrow$ suppression of error $p \rightsquigarrow 3p^2 - 2p^3$, as classically.

By linearity, this also works for part of a larger entangled state:

$$\alpha|0\rangle|a\rangle + \beta|1\rangle|b\rangle \xmapsto{\text{encode}} \alpha|000\rangle|a\rangle + \beta|111\rangle|b\rangle$$

$$\xmapsto[X_1]{\text{error:}} \alpha|100\rangle|a\rangle + \beta|011\rangle|b\rangle \xmapsto[\text{Correct: } X_1]{\text{meas.: } P_1} \alpha|000\rangle|a\rangle + \beta|111\rangle|b\rangle$$

What about <u>continuous errors</u>, e.g.

$$|\hat\psi\rangle \longmapsto e^{i\vartheta X_i}|\hat\psi\rangle = (\cos\vartheta\, I + i\sin\vartheta\, X_i)|\hat\psi\rangle \,?$$

$$|\hat\psi\rangle = \alpha|000\rangle + \beta|111\rangle \xrightarrow[\text{e.g. } X_3]{\text{error,}} \alpha(\cos\vartheta|000\rangle + i\sin\vartheta|001\rangle) + \beta(\cos\vartheta|111\rangle + i\sin\vartheta|110\rangle)$$

$$= \cos\vartheta\,\underbrace{(\alpha|000\rangle + \beta|111\rangle)}_{\text{syndrome } P_0} + i\sin\vartheta\,\underbrace{(\alpha|001\rangle + \beta|110\rangle)}_{\text{syndrome } P_3}$$

prob.: $|\cos\vartheta|^2$ prob.: $|\sin\vartheta|^2$

<u>Syndrome measurement collapses state</u> into:

$p = \cos^2\vartheta$: result P_0,

post-meas. state $\alpha|000\rangle + \beta|111\rangle$,

$0 \equiv$ no correction : OK ✓

$p = \sin^2\vartheta$: result P_3,

post-meas. state $\alpha|001\rangle + \beta|110\rangle$,

$3 \equiv$ correction: flip bit 3 :

$\Longrightarrow \alpha|000\rangle + \beta|111\rangle$: OK ✓

Measurement of error syndrome $\{P_\alpha\}$ collapses continuous error onto one of the 4 correctable discrete errors:

- measurement "digitalizes" error
- sufficient to study discrete (distinguishable) errors (will be formalized later)

A different perspective on syndrome measurement & correction (the "stabilizer formalism" — more later):

$|000\rangle, |111\rangle$: +1 eigenstates of $Z_1 Z_2$ and $Z_2 Z_3$ ("stabilizers")

Measure $\underbrace{Z_1 Z_2}$ and $\underbrace{Z_2 Z_3}$:

compare qubits 1&2 and 2&3

$(+1,+1)$: no error

$(-1,+1)$: qubit 1 flipped

$(+1,-1)$: qubit 3 flipped

$(-1,-1)$: qubit 2 flipped

More formally:

encoded state $|\hat{\varphi}\rangle = \alpha|000\rangle + \beta|111\rangle$:

$$\Longrightarrow Z_1 Z_2 |\hat{\varphi}\rangle = |\hat{\varphi}\rangle \ , \ Z_2 Z_3 |\hat{\varphi}\rangle = |\hat{\varphi}\rangle$$

Bit flip error, e.g. X_1:

X_1 anti-commutes with $Z_1 Z_2$

$$\Longrightarrow \langle\hat{\psi}| X_1 Z_1 Z_2 X_1 |\hat{\psi}\rangle = -\langle\hat{\psi}| Z_1 Z_2 |\hat{\psi}\rangle$$

$$= -1$$

Thus:

Outcome -1 for $Z_1 Z_2$ $\Longleftrightarrow$ an error which anti-commutes with $Z_1 Z_2$ has occurred.

The correction operation must satisfy the <u>same</u> anti-commutation relations (and some further properties) $\rightarrow$ <u>later</u>!

Have focused on X errors.

But what about Z errors?

$$\alpha|000\rangle + \beta|111\rangle \xmapsto[\text{on qubit 1}]{Z \text{ error}} \alpha|000\rangle - \beta|111\rangle$$

This is still a state in the code space

(i.e., a valid encoded state $|\hat{\psi}\rangle$)

$\Rightarrow$ error not detectable, but it has changed

$|\hat{\psi}\rangle$. After decoding, the error acts as

$$\alpha|0\rangle + \beta|1\rangle \longmapsto \alpha|0\rangle - \beta|1\rangle,$$

i.e. as a logical Z operator.

"logical operator" =
operator on encoded qubit.

$\Rightarrow$ 3-qubit bit flip code cannot protect
against single "phase flip error" Z_i.

Stabilizer picture:

Error Z_i commutes with stabilizers Z_1Z_2 & Z_2Z_3

$\Longrightarrow$ it cannot be detected.

But: Z_i cannot be expressed as a product of the stabilizers Z_1Z_2 & Z_2Z_3 $\Longrightarrow$ Logical error!

c) The 3-qubit phase-flip code

Can we correct against Z errors?

$$Z|+\rangle = |-\rangle, \quad Z|-\rangle = |+\rangle$$

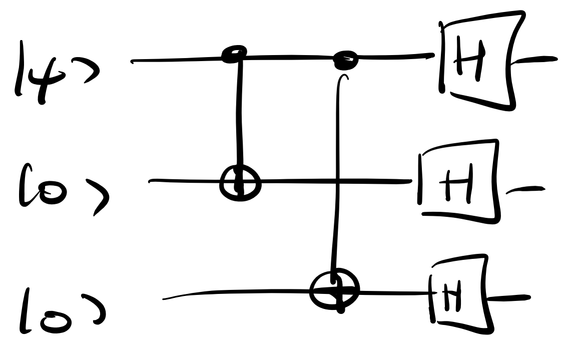
$\Longrightarrow$ Z error $\hat{=}$ bit flip error in $|\pm\rangle$-basis.

Use encoding $\alpha|0\rangle + \beta|1\rangle \longmapsto \alpha|\hat{0}\rangle + \beta|\hat{1}\rangle$,

with $|\hat{0}\rangle := |{+}{+}{+}\rangle$, $|\hat{1}\rangle = |{-}{-}{-}\rangle$:

Will protect against single Z errors!

Encoding:



Syndrome measurement:

$$\tilde{P}_\alpha := H^{\otimes 3} P_\alpha H^{\otimes 3}$$

(or via stabilizers $X_1 X_2$ & $X_2 X_3$).

Recovery operation:

$$H X_i H = Z_i$$

(anti-com. with stabilizers).

Problem:

Now, there is no protection against bit flip errors X_i

— and X_i acts as a logical Z operator!