3. The Quantum Error Correction Conditions

Definition: Given $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$, a Quantum Error Correction Code (QECC) on $\mathcal{H}$ is a subspace $\mathcal{C} \subset \mathcal{H}$ (the code space, with $|\psi\rangle \in \mathcal{C}$ codewords).

We denote by $|\hat{i}\rangle$ an (arbitrary, but fixed) basis of $\mathcal{C}$.

Definition: A noise model on $\mathcal{H}$ is a CP map

$$\mathcal{E}(\rho) = \sum E_\alpha \rho E_\alpha^\dagger \; ; \quad \sum E_\alpha^\dagger E_\alpha \leq \underline{\underline{I}} \; !$$

(i.e., error E_α occurs w/ prob. $\mathrm{tr}(E_\alpha^\dagger E_\alpha \rho)$,

e.g. $E_\alpha \propto$ single-qubit Paulis.)

Note: This is only the part of the noise which we want to correct – thus $\sum E_\alpha^\dagger E_\alpha \leq I$. The total noise is

$$\mathcal{N}(\rho) = \mathcal{E}(\rho) + \underbrace{\sum N_\gamma \rho N_\gamma^\dagger}_{\text{not correctable noise.}} , \quad \sum E_\alpha^\dagger E_\alpha + \sum N_\gamma^\dagger N_\gamma = I .$$

Definition: We say that a QECC $\mathcal{C}$ can correct for an error $\mathcal{E}$ if there exists a recovery map $\mathcal{R}$, i.e. a CP map $\mathcal{R}$ such that

$$\mathcal{R}(\mathcal{E}(\rho)) \propto \rho \qquad \forall \rho = |\psi\rangle\langle\psi|, \ |\psi\rangle \in \mathcal{C}$$

Note: $\mathcal{R}$ must correct the error deterministically, i.e., $\mathcal{R}$ must be trace-preserving on states supported on the image of $\mathcal{C}$ under $\mathcal{E}$, i.e., on states obtained by noise from a code state.)

Theorem (Quantum Error Correction Condition):

Given $\mathcal{C}$ and $\mathcal{E}(\cdot) = \sum E_\alpha \cdot E_\alpha^\dagger$, there exists a recovery $\mathcal{R}$ (i.e. $\mathcal{C}$ can correct for $\mathcal{E}$) if and only if

$$\boxed{\langle \hat{i} | E_\alpha^\dagger E_\beta | \hat{j} \rangle = c_{\alpha\beta}\, \delta_{ij}} \qquad (*)$$

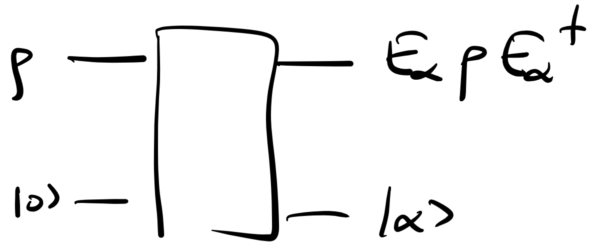
for some ONB $\{|\hat{i}\rangle\}$ $(\langle \hat{i}|\hat{j}\rangle = \delta_{ij})$ of $\mathcal{C}$.

Intuition:

① Orthogonal states remain orthogonal (R cannot make states *more* orthogonal!)

② Environment learns nothing about state:

Stinespring:



$$\mathrm{prob}(\alpha) = \left(\sum \bar{a}_i \langle i|\right) E_\alpha^\dagger E_\alpha \left(\sum a_j |j\rangle\right)$$

$$= \underbrace{\sum |a_i|^2}_{=1} c_{\alpha\alpha} = c_{\alpha\alpha} \text{ indep. of state.}$$

Proof:

'existence of $R \Longrightarrow (*)$":

Lemma: $\sum_\tau K_\tau |\psi\rangle\langle\psi| K_\tau^\dagger \propto |\psi\rangle\langle\psi| \quad \forall |\psi\rangle \in \mathcal{C}$

$\Longrightarrow K_\tau |\psi\rangle = a_\tau |\psi\rangle$

with a_τ indep. of $|\psi\rangle$.

Proof: $\sum_\tau K_\tau |\psi\rangle\langle\psi| K_\tau^\dagger \propto |\psi\rangle\langle\psi|$

Choose any $|\chi\rangle$ s.th. $\langle\chi|\psi\rangle = 0$

$$\Rightarrow \sum_\tau \underbrace{\langle\chi|K_\tau|\psi\rangle\langle\psi|K_\tau^\dagger|\chi\rangle}_{\geq 0} \propto \langle\chi|\psi\rangle\langle\psi|\chi\rangle = 0$$

$$\Rightarrow \langle\chi|K_\tau|\psi\rangle = 0 \quad \forall \tau$$

$$\Rightarrow K_\tau|\psi\rangle = a_\tau(|\psi\rangle)\,|\psi\rangle.$$

What if $a_\tau(|\psi\rangle)$ dep. on $|\psi\rangle$? Choose $|\psi_1\rangle, |\psi_2\rangle$ s.th. $a_\tau(|\psi_1\rangle) \neq a_\tau(|\psi_2\rangle)$. Then,

$$K_\tau(|\psi_1\rangle + |\psi_2\rangle) = a_\tau(|\psi_1\rangle)|\psi_1\rangle + a_\tau(|\psi_2\rangle)|\psi_2\rangle$$

$$\not\propto |\psi_1\rangle + |\psi_2\rangle \quad ↯$$

$$\Rightarrow a_\tau(|\psi\rangle) \equiv a_\tau$$

$$\Rightarrow K_\tau|\psi\rangle = a_\tau|\psi\rangle. \qquad \blacksquare$$

Let $\mathcal{R}(\cdot) = \sum R_\gamma \cdot R_\gamma^\dagger$.

Then: $\mathcal{R}(\mathcal{E}(|\psi\rangle\langle\psi|)) \propto |\psi\rangle\langle\psi| \quad \forall |\psi\rangle \in \mathcal{C}$

Lemma

$$\Longrightarrow \quad R_\gamma E_\alpha |\psi\rangle = a_{\gamma\alpha} |\psi\rangle \quad \forall |\psi\rangle \in \mathcal{C}$$

ONB $|\hat{i}\rangle, |\hat{j}\rangle$:

$$\Longrightarrow \sum_\gamma \langle \hat{i} | E_\alpha^\dagger R_\gamma^\dagger R_\gamma E_\beta | \hat{j} \rangle = \sum_\gamma \overline{a_{\gamma\alpha}} a_{\gamma\beta} \langle \hat{i} | \hat{j} \rangle$$

$$=: c_{\alpha\beta} \, \delta_{ij}$$

$$\Longrightarrow \langle \hat{i} | E_\alpha^\dagger \underbrace{\Big(\sum_\gamma R_\gamma^\dagger R_\gamma \Big)}_{= I \text{ on image of } \mathcal{C} \text{ under } \mathcal{E}.} E_\beta | \hat{j} \rangle = c_{\alpha\beta} \, \delta_{ij}$$

$$\Longrightarrow \langle \hat{i} | E_\alpha^\dagger E_\beta | \hat{j} \rangle = c_{\alpha\beta} \, \delta_{ij} \qquad \blacksquare$$

"$\circledast \Rightarrow$ existence of $\mathcal{R}$":

Construct explicit recovery channel $\mathcal{R}(\cdot) = \sum R_\gamma \cdot R_\gamma^\dagger$.

Step 1: Use gauge degree of freedom in E_α:

$$\mathcal{E}(\rho) = \sum E_\alpha \rho E_\alpha^\dagger = \sum F_\beta \rho F_\beta^\dagger$$

if & only if $F_\beta = \sum_\alpha V_{\beta\alpha} E_\alpha$, V isometry.

Choose V unitary s.th. $\displaystyle\sum_{\alpha\beta} \overline{V_{\varepsilon\alpha}} \, c_{\alpha\beta} \, V_{\tau\beta} = \lambda_\varepsilon \, \delta_{\varepsilon\tau}$ diagonal

$$\circledast \Rightarrow \langle \hat{i} | F_\varepsilon^\dagger F_\tau | \hat{j} \rangle = \sum_{\alpha,\beta} \langle \hat{i} | \overline{V_{\varepsilon\alpha}} E_\alpha^\dagger E_\beta V_{\tau\beta} | \hat{j} \rangle$$

$$= \sum_{\alpha,\beta} \overline{V_{\varepsilon\alpha}} V_{\tau\beta} \langle \hat{i} | E_\alpha^\dagger E_\beta | \hat{j} \rangle$$

$$= \sum_{\alpha\beta} \overline{V_{\varepsilon\alpha}} V_{\tau\beta} \, c_{\alpha\beta} \, \delta_{ij}$$

$$= \lambda_\varepsilon \delta_{\varepsilon\tau} \delta_{ij}$$

$\Rightarrow$ Different errors F_ε can be distinguished by a projective measurement!

Note that $\sum_\varepsilon \lambda_\varepsilon = \sum_\varepsilon \underbrace{\langle \hat{i} | F_\varepsilon^\dagger F_\varepsilon | \hat{i} \rangle}_{= \lambda_\varepsilon \text{ : prob. of error } \varepsilon} \leq \langle \hat{i} | I | \hat{i} \rangle = 1.$

Step 2: Measure ε and undo error F_ε.

Want R_γ s.th. $R_\gamma F_\varepsilon | \hat{i} \rangle = \sqrt{\lambda_\varepsilon} \delta_{\gamma\varepsilon} | \hat{i} \rangle$!

prob. of error F_ε.

Choose $R_\gamma := \frac{1}{\sqrt{\lambda_\varepsilon}} \sum_j | \hat{j} \rangle \langle \hat{j} | F_\gamma^\dagger$.

If $\lambda_\varepsilon = 0$, then $R_\gamma = 0$ is a solution.

$$\Rightarrow R_\gamma F_\varepsilon | \hat{i} \rangle = \frac{1}{\sqrt{\lambda_\varepsilon}} \sum_j | \hat{j} \rangle \underbrace{\langle \hat{j} | F_\gamma^\dagger F_\varepsilon | \hat{i} \rangle}_{= \lambda_\varepsilon \delta_{\gamma\varepsilon} \delta_{ij}} = \sqrt{\lambda_\varepsilon} \delta_{\gamma\varepsilon} | \hat{i} \rangle.$$

$$\Rightarrow R_\gamma F_\varepsilon |\hat{\varphi}\rangle = \sqrt{\lambda_\varepsilon}\, \delta_{\gamma\varepsilon} |\hat{\varphi}\rangle \quad \forall |\hat{\varphi}\rangle \in \mathcal{C}$$

$$\Rightarrow \mathcal{R}(\mathcal{E}(|\hat{\varphi}\rangle\langle\hat{\varphi}|)) = \sum_{\gamma,\varepsilon} R_\gamma F_\varepsilon |\hat{\varphi}\rangle\langle\hat{\varphi}| F_\varepsilon^\dagger R_\gamma^\dagger$$

$$= \sum_\varepsilon \lambda_\varepsilon |\hat{\varphi}\rangle\langle\hat{\varphi}| \propto |\hat{\varphi}\rangle\langle\hat{\varphi}| \quad \forall |\hat{\varphi}\rangle \in \mathcal{C},$$

and $\mathrm{tr}\left(\mathcal{R}(\mathcal{E}(|\hat{\varphi}\rangle\langle\hat{\varphi}|))\right) = \sum \lambda_\varepsilon =$

$$= \sum \langle\hat{\varphi}| F_\varepsilon^\dagger F_\varepsilon |\hat{\varphi}\rangle = \mathrm{tr}\left(\mathcal{E}(|\hat{\varphi}\rangle\langle\hat{\varphi}|)\right),$$

i.e. $\mathcal{R}$ is trace-preserving on the image of $\mathcal{C}$ under $\mathcal{E}$. ∎

Definition: Single-qubit errors correspond to an error model with noise operators of the form

$$E_\alpha = \sum_{k,s} \omega_{\alpha,k,s}\, \sigma_s^k \longleftarrow \text{k'th Pauli matrix on qubit s.}$$

Observation: A QECC can correct for any single-qubit error if it can correct for any single-qubit Pauli error.

Proof: Code can correct for any single Pauli error $\Rightarrow$

$$\langle\hat{i}| \sigma_s^k \sigma_r^l |\hat{j}\rangle = c_{skrl}\, \delta_{ij} \Rightarrow$$

$\Rightarrow \langle\hat{i}| E_\alpha^\dagger E_\beta |\hat{j}\rangle = \tilde{c}_{\alpha\beta}\, \delta_{ij} \Rightarrow$ can correct of any single-qubit error. ∎

In particular: A QECC which can correct for single-qubit depolarizing noise

$$\mathcal{E}(\rho) = (1-p)\rho + \frac{p}{3}\left(X\rho X + Y\rho Y + Z\rho Z\right)$$

on any one of k qubits – i.e. a noise

$$\mathcal{E}(\rho) = (1-kp)\,\rho + \sum_{i=1}^{k} \frac{p}{3}\left(X_i\rho X_i + Y_i\rho Y_i + Z_i\rho Z_i\right)$$

is also robust against any single-qubit error!

Corollary: To check for robustness against arbitrary single-qubit errors, it is sufficient to check the error model with

$$\{E_\alpha\} \propto \{I, X_1, X_2, \dots, Y_1, Y_2, \dots, Z_1, Z_2, \dots\}$$

E_α up to prefactors

The analogous result holds for k-qubit errors vs. k-qubit Paulis.

Exercise suggestion: Check q. error correction condition for 3-qubit & 9-qubit code!