Lecture & Proseminar 250078/250042
"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2022/23

— Exercise Sheet #1 —

Problem 1: Pauli matrices.
Recall the Pauli matrices from the lecture, which in the computational basis $\{|0\rangle, |1\rangle\}$ are of the form

$$X = \sigma_x = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad Y = \sigma_y = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

1. Show that the Pauli matrices are all hermitian, unitary, square to the identity, and different Pauli matrices anticommute.
2. Check the relation $\sigma_\alpha \sigma_\beta = \sum_\gamma i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma + \delta_{\alpha\beta} I$ $(\alpha, \beta, \gamma = 1, 2, 3)$, with $\varepsilon_{\alpha\beta\gamma}$ the fully antisymmetric tensor (i.e. $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$, $\varepsilon_{321} = \varepsilon_{213} = \varepsilon_{132} = -1$, and zero otherwise).
3. The *trace* $\mathrm{tr}[X]$ is defined as the sum of the diagonal elements of X, i.e., $\mathrm{tr}[X] := \sum_i X_{ii}$. Determine $\mathrm{tr}[I]$, $\mathrm{tr}[\sigma_\alpha]$, and $\mathrm{tr}[\sigma_\alpha \sigma_\beta]$.
4. Write each operator X, Y and Z using bra-ket notation with states from the computational basis.
5. Find the eigenvalues e_i and eigenvectors $|v_i\rangle$ of the Pauli matrices (expressed in the computational basis), and write them in their diagonal form $e_1|v_0\rangle\langle v_0| + e_1|v_1\rangle\langle v_1|$.
6. Determine the measurement operators $\{E_i\}$ corresponding to a measurement of the Y observable. For a state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, determine the probabilities for the different outcomes for a measurement of the Y observable, and find the corresponding post-measurement states.
7. Write all tensor products of Pauli matrices $\sigma_\alpha \otimes \sigma_\beta$ (including the identity $\sigma_0 = I$) as 4×4 matrices.

Problem 2: Matrix spaces as Hilbert spaces.
Let $\mathcal{V}_d$ be the space of all complex $d \times d$ matrices, and $\mathcal{W}_d \subset \mathcal{V}_d$ the space of all hermitian complex $d \times d$ matrices (i.e. for $M \in \mathcal{W}_d$, $M = M^\dagger$).

1. Show that $\mathcal{V}_d$ forms a vector space over $\mathbb{C}$, and $\mathcal{W}_d$ forms a vector space over $\mathbb{R}$, but not over $\mathbb{C}$. We will in the following always consider $\mathcal{V}_d$ as a complex and $\mathcal{W}_d$ as a real vector space.
2. Show that the Pauli matrices together with the identity, $\Sigma := \{\sigma_i\}_{i=0}^3$, form a basis for both $\mathcal{V}_2$ (over $\mathbb{C}$) and $\mathcal{W}_2$ (over $\mathbb{R}$).
3. Show that
$$(A, B) = \mathrm{tr}[A^\dagger B]$$
defines a scalar product (the "Hilbert-Schmidt scalar product") both for $\mathcal{V}_d$ and for $\mathcal{W}_d$. Here, $\mathrm{tr}[X]$ is the trace, i.e., the sum of the diagonal elements.
4. Show that the Pauli matrices Σ form an orthonormal basis (ONB) with respect to the suitably rescaled Hilbert-Schmidt scalar product.
5. Use the fact that for any scalar product $(\vec{v}, \vec{w})$ and a corresponding ONB $\vec{w}_i$, we can write
$$\vec{v} = \sum_i \vec{w}_i (\vec{w}_i, \vec{v}) \ ,$$
to express a general matrix in $M \in \mathcal{V}_2$ as
$$M = \sum m_i \sigma_i \ .$$
What is the form of the m_i? What special property do the m_i satisfy for $M \in \mathcal{W}_2$?

6. Show that a hermitian orthonormal basis also exists for $\mathcal{V}_d$ and $\mathcal{W}_d$. (Ideally, explicitly construct such a basis.)

Problem 3: Unitary invariance and Bell states.

1. Show that the singlet state
$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle_{AB} - |10\rangle_{AB})$$
is invariant under joint rotations by the same 2×2 unitary U, i.e.,
$$|\Psi^-\rangle = (U \otimes U)|\Psi^-\rangle$$
for any special unitary matrix $U \in \mathrm{SU}(2)$, i.e. $U^\dagger U = I$, $\det(U) = 1$. How does this formula change when $\det(U) \neq 1$?

2. Show that this implies that if we measure the spin in any direction $\vec{v}$, $|\vec{v}| = 1$ – this measurement is described by the measurement operator $S_{\vec{v}} = \sum_{i=1}^{3} v_i \sigma_i$, i.e. the projectors onto its eigenvectors – we obtain perfectly random and opposite outcomes.
(*Hint:* An elegant way of doing so is to first show that any $S_{\vec{v}}$ has the same eigenvalues as the Z matrix and therefore can be rotated to it, i.e., there exists a $U_{\vec{v}}$ s.th. $U_{\vec{v}} S_{\vec{v}} U_{\vec{v}}^\dagger = Z$. Note that there are very elegant ways to show that the eigenvalues are ± 1 as well!)

3. Determine the states
$$\begin{aligned} &(X \otimes I)|\Psi^-\rangle\,, \qquad (I \otimes X)|\Psi^-\rangle\,, \\ &(Y \otimes I)|\Psi^-\rangle\,, \qquad (I \otimes Y)|\Psi^-\rangle\,, \\ &(Z \otimes I)|\Psi^-\rangle\,, \qquad (I \otimes Z)|\Psi^-\rangle\,. \end{aligned}$$
In the light of point 1, why are they pairwise equal (up to global phases)?
Note: Together with $|\Psi^-\rangle$, these are known as the four *Bell states*.

4. Show that the maximally entangled state
$$|\Omega\rangle = \sum_{i=1}^{d} |i, i\rangle$$
of two qu-d-its (i.e., systems with a Hilbert space $\mathbb{C}^d$) is invariant under $U \otimes \bar{U}$, where U is any $d \times d$ unitary, that is,
$$|\Omega\rangle = (U \otimes \bar{U})|\Omega\rangle\,.$$