Lecture & Proseminar 250078/250042
"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2022/23

— Exercise Sheet #2 —

Problem 4: Bloch sphere for pure states.

a) Recall (or rederive) the eigenstates (=eigenvectors) and eigenvalues of the Pauli matrices. Determine the angles θ and ϕ of those eigenstates on the Bloch sphere, and depict their position on the Bloch sphere.

b) Given a state

$$|\psi\rangle = e^{i\chi}\big[\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle\big] \tag{1}$$

show that

$$|\psi\rangle\langle\psi| = \tfrac{1}{2}(I + \vec{v}\cdot\vec{\sigma}) \ \text{ with } \vec{v}\in\mathbb{R}^3 \text{ and } |\vec{v}| = 1\ , \tag{2}$$

(i.e., $\vec{v}$ is a vector on the unit sphere in $\mathbb{R}^3$), where $\vec{v}\cdot\vec{\sigma} = \sum_{i=1}^{3} v_i\sigma_i$. (You should find that $\vec{v}$ is exactly the point on the Bloch sphere with spherical coordinates in θ and ϕ, just as introduced in the lecture.)

c) Show that the expectation value of the Pauli operators is $\langle\psi|\sigma_i|\psi\rangle = v_i$; i.e., $|\psi\rangle$ desribes a spin which is polarized along the direction $\vec{v}$.

d) Show that for any state $|\psi\rangle$ with corresponding Bloch vector $\vec{v}$, the state $|\phi\rangle$ orthogonal to it, i.e. with $\langle\psi|\phi\rangle = 0$ (for qubits, i.e., in $\mathbb{C}^2$, this state is uniquely determined up to a phase!), is described by the Bloch vector $-\vec{v}$, i.e., it is located at the opposite point of the Bloch sphere.
(*Bonus question:* Derive a general expression for the overlap $|\langle\phi|\psi\rangle|^2$ of two arbitrary states in terms of the corresponding Bloch vectors.)

(*Note:* A particularly elegant way to check c) and d) is to use that $\langle\psi|O|\psi\rangle = \mathrm{tr}[|\psi\rangle\langle\psi|\,O]$ together with Eq. (2) and $\mathrm{tr}[\sigma_i\sigma_j] = 2\delta_{ij}$, but the results can of course also be derived directly from Eq. (1) with a bit more brute force.)

Problem 5: Bloch sphere for mixed states.

a) Show that any hermitian matrix ρ with $\mathrm{tr}\,\rho = 1$ can be written as

$$\rho = \frac{I + \vec{r}\cdot\vec{\sigma}}{2}$$

with a vector $\vec{r}\in\mathbb{R}^3$.

b) What are the eigenvalues of ρ? How do they depend on $|\vec{r}|$?

c) What property do the points $\vec{r}$ for which $\rho \geq 0$ satisfy?

d) Interpret this in terms of the Bloch sphere: What do points on the inside or outside of the Bloch sphere correspond to? What about the surface of the sphere?

e) What is the interpretation of the point at the center of the Bloch sphere? What is the interpretation of points along the z axis? How does this generalize to other points inside the Bloch sphere?

f) What is the location of a state $\rho = p|\psi\rangle\langle\psi| + (1-p)|\phi\rangle\langle\phi|$ in the Bloch sphere? How does this generalize to general convex combinations $\rho = \sum p_i|\psi_i\rangle\langle\psi_i|$?

Problem 6: No-cloning theorem.

a) Show that it is possible to build a cloning device which can copy all computational basis states $\{|i\rangle\}$, i.e., a unitary U such that

$$U|i\rangle|0\rangle = |i\rangle|i\rangle \ .$$

Give an explicit construction of such a U for qubits.

b) Show that such a cloner U can also be built for any other ONB $\{|\phi_i\rangle\}$,

$$U|\phi_i\rangle|0\rangle = |\phi_i\rangle|\phi_i\rangle \ .$$

c) The no-cloning theorem states that there exists no unitary U which implements $U|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle$ for all $|\psi\rangle$. Show that this even holds when we allow for an additional auxiliary system, i.e. there exists no U which implements

$$U|\psi\rangle|0\rangle|0\rangle = |\psi\rangle|\psi\rangle|\gamma_\psi\rangle$$

for any final state $|\gamma_\psi\rangle$ of the auxiliary system (which can depend on $|\psi\rangle$ in any possible way).

Problem 7: Ensemble decompositions by measurement.

a) Consider a state $|\psi\rangle = \alpha|0\rangle_A|0\rangle_B + \beta|1\rangle_A|1\rangle_B$, shared between two parties A and B, with Hilbert space dimensions $d_A = 2$ and $d_B = 4$, respectively. Determine the probabilities p_i and Alice's post-measurement states $|\phi_i\rangle$ if Bob measures in the basis (check that it is an ONB!)

$$(|0\rangle + |2\rangle)/\sqrt{2}\,, \quad (|1\rangle + |3\rangle)/\sqrt{2}\,, \quad (|0\rangle \pm |1\rangle - |2\rangle \mp |3\rangle)/2$$

(note the $\pm$).

What ensemble interpretation of Alice's state does this give? Check that this gives the correct reduced density matrix.

b) Consider the case where Bob's system has a general dimension d_B, and where he measures in a basis

$$|b_i\rangle = \sum u_{ij}|j\rangle \ .$$

i) What properties does the matrix $U = (u_{ij})$ satisfy?

ii) What is the form of the resulting post-measurement ensemble $\{(p_i, |\phi_i\rangle)\}$ for Alice's state?