Lecture & Proseminar 250078/250042
"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2022/23

— Exercise Sheet #4 —

Problem 11: Schmidt decomposition and singular value decomposition

Part 1: Schmidt decomposition

Find the Schmidt decomposition of the following states:

$$|\psi_1\rangle = \frac{|0\rangle|0\rangle + |1\rangle|1\rangle}{\sqrt{2}}$$
$$|\psi_2\rangle = \frac{|0\rangle|0\rangle + |0\rangle|1\rangle}{\sqrt{2}}$$
$$|\psi_3\rangle = \frac{|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle}{2}$$
$$|\psi_4\rangle = \frac{|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle - |1\rangle|1\rangle}{2}$$
$$|\psi_5\rangle = \frac{|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle}{\sqrt{3}}$$

Part 2: SVD from eigenvalue decomposition

In the following, we will construct the singular value decomposition (SVD) from the eigenvalue decomposition.
To this end, consider a general rectangular matrix M of size $m \times n$, $m \leq n$.

1. Consider the eigenvalue decomposition of $MM^\dagger$, $MM^\dagger = U\Lambda U^\dagger$, with U unitary and Λ diagonal. What property do the eigenvalues, i.e. the entries of Λ, satisfy?
2. Define $D = \sqrt{\Lambda}$ (since Λ is diagonal, this is the square root of the diagonal elements). Consider first the case where all diagonal elements of D are non-zero. Let
$$V := M^\dagger U D^{-1} \ .$$
   (a) What is $V^\dagger V$?
   (b) What is $UDV^\dagger$?
3. Generalize this argument to the case where $MM^\dagger$ has eigenvalues which are zero.

Problem 12: Quantum channels.

In this problem, we will study some commonly appearing quantum channels. In addition to the problems listed, verify for each channel that it is a CPTP map (completely positive trace preserving map) and give its Kraus representation.

1. *Dephasing channel.* This channel acts as
$$\mathcal{E}(\rho) = (1-p)\,\rho + p\,Z\rho Z \ .$$
   Show that the action of the dephasing channel on the Bloch vector is
$$(r_x, r_y, r_z) \mapsto ((1-2p)r_x, (1-2p)r_y, r_z) \ ,$$
   i.e., it preserves the component of the Bloch vector in the Z direction, while shrinking the X and Y component.

2. *Amplitude damping channel.* The amplitude damping channel is giving by the Kraus operators

$$M_0 = \sqrt{\gamma}|0\rangle\langle 1|, \quad M_1 = |0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1| \ ,$$

where $0 \le \gamma \le 1$. Here, M_0 describes a decay from $|1\rangle$ to $|0\rangle$, and γ corresponds to the decay rate.

   (a) Consider a single-qubit density operator with the following matrix representation with respect to the computation basis

$$\rho = \begin{pmatrix} 1-p & \eta \\ \eta^* & p \end{pmatrix},$$

   where $0 \le p \le 1$ and η is some complex number. Find the matrix representation of this density operator after the action of the amplitude damping channel.

   (b) Show that the amplitude damping channel obeys a composition rule. Consider an amplitude damping channel $\mathcal{E}_1$ with parameter γ_1 and consider another amplitude damping channel $\mathcal{E}_2$ with parameter γ_2. Show that the composition of the channels, $\mathcal{E} = \mathcal{E}_1 \circ \mathcal{E}_2$, $\mathcal{E}(\rho) = \mathcal{E}_1(\mathcal{E}_2(\rho))$, is an amplitude damping channel with parameter $1-(1-\gamma_1)(1-\gamma_2)$. Interpret this result in light of the interpretation of the γ's as a decay probability.

3. *Twirling operation.* Twirling is the process of applying a random Pauli operator (including the identity) with equal probability. Explain why this corresponds to the channel

$$\mathcal{E}(\rho) = \tfrac{1}{4}\rho + \tfrac{1}{4}X\rho X + \tfrac{1}{4}Y\rho Y + \tfrac{1}{4}Z\rho Z \ .$$

   Show that the output of this channel is the maximally mixed state for any input, $\mathcal{E}(\rho) = \frac{1}{2}I$.

   Hint: Represent the density operator as $\rho = \frac{1}{2}(I + r_x X + r_y Y + r_z Z)$ and apply the commutation rules of the Pauli operators.