**Lecture & Proseminar 250078/250042**
**"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2022/23**

**— Exercise Sheet #5 —**

**Problem 13: CHSH inequality – Tsirelson's bound.**
Tsirelson's inequality bounds the largest possible violation of the CHSH inequality $|\langle C\rangle| \leq 2$, with

$$\langle C\rangle = \langle a_0 b_0\rangle + \langle a_1 b_0\rangle + \langle a_0 b_1\rangle - \langle a_1 b_1\rangle \ . \tag{1}$$

in quantum mechanics – namely $|\langle C\rangle| = 2\sqrt{2}$). To this end, let $a_0, a_1, b_0, b_1$ be Hermitian operators (on some finite-dimensional complex Hilbert space) with eigenvalues $\pm 1$, so that

$$a_0^2 = a_1^2 = b_0^2 = b_1^2 = I \ .$$

Here, $a_0$ and $a_1$ describe the two measurements of Alice, and $b_0$ and $b_1$ those of Bob; in particular, this means that Alice's and Bob's measurements commute, i.e. $[a_x, b_y] = 0$ for all $x, y = 0, 1$. Define

$$C = a_0 b_0 + a_1 b_0 + a_0 b_1 - a_1 b_1 \ .$$

1. Determine $C^2$.
2. The operator norm of a bounded operator $M$ is defined by

$$\|M\| = \sup_{|\psi\rangle} \frac{\|M|\psi\rangle\|}{\||\psi\rangle\|} \ .$$

   Verify that the operator norm has the properties

$$\begin{aligned} \|MN\| &\leq \|M\|\|N\| \ , \\ \|M+N\| &\leq \|M\| + \|N\| \ . \end{aligned} \tag{2}$$

3. Show that $\|M\|$ equals the maximum eigenvalue of $\sqrt{M^\dagger M}$ (and thus, the largest singular value of $M$). What is then the operator norm of a hermitian operator in terms of its eigenvalues?
4. Find an upper bound on the norm $\|C^2\|$ (using the inequalities (2)).
5. Show that for Hermitian operators $\|C^2\| = \|C\|^2$. Use this to obtain an upper bound on $\|C\|$.
6. Explain how this inequality gives a bound on the maximum possible violation of the CHSH inequality in quantum mechanics. This is known as Tsirelson's bound, or Tsirelson's inequality.

**Problem 14: Teleportation-insired protocols.**

In this problem, we will get to know two variants of the teleportation protocol.

***Part 1: Gate teleportation.***

Gate teleportation is a variation of quantum teleportation that is being used in fault-tolerant quantum computation (a topic which will be covered later in the course of the lecture).
Suppose that we would like to perform a single-qubit gate (i.e., unitary) $U$ on a qubit in state $|\psi\rangle$, but the gate is difficult to perform – e.g., it might fail and thereby destroy the state on which we act on. On the other hand, the gate $U\sigma_j U^\dagger$, where $\sigma_j$ is any one of the three Pauli matrices, is easy to perform.

1. Verify that such a situation is given when the difficult operation is $U = \left(\begin{smallmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{smallmatrix}\right)$, while Paulis and $S = \left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right)$ are easy to realize.
2. Consider the following protocol to implement $U$ on a state $|\psi\rangle_{A'}$:

- Prepare $|\chi\rangle_{AB} = (I_A \otimes U_B)|\Phi^+\rangle_{AB}$, with $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. ($U_B$ is still hard to implement, but we can try as many times as we want without breaking anything.)
- Perform a measurement of $A'A$ in the Bell basis ($A'$ is the register used to store $|\psi\rangle_{A'}$).
- Depending on the measurement outcome, apply $U\sigma_j U^\dagger$ on the $B$ system.

Show that this protocol works as it should – that is, it yields the state $U|\psi\rangle$ in the $B$ register with unit probability.

***Part 2: Remote state preparation.***

Remote state preparation is another variation on the teleportation protocol. In the variant we consider here, Alice has a *classical description* of a state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle)$ (on the equator of the Bloch sphere), i.e., she knows $\phi$. The task is to prepare the state $|\psi\rangle$ on Bob's side, without Bob learning anything about $\phi$.
To this end, let Alice and Bob share a maximally entangled state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

1. Find a state $|\chi\rangle$ such that when Alice's part of $|\Phi^+\rangle$ is projected onto $|\chi\rangle$, Bob is left with $|\psi\rangle$.
2. Now let Alice perform a measurement in the basis $\{|\chi\rangle, |\chi^\perp\rangle\}$, where $|\chi^\perp\rangle$ is the state perpendicular to $|\chi\rangle$ (since the space is 2-dimensional, $|\chi^\perp\rangle$ is unique up to a phase). Determine the post-measurement state of Bob for both of Alice's outcomes.
3. Show that if Alice communicates one bit to Bob, and Bob performs an operation which depends on this bit (which information is in the bit? what operation does Bob have to perform?), then Bob recovers $|\psi\rangle$ with unit probability.
4. A more "direct" way – given we know the protocol for teleportation – for Alice and Bob to realize the remote state preparation protocol would have been that Alice prepares $|\psi\rangle$ and then teleports it to Bob. Is there a way to relate these two protocols? How can the remote state preparation protocol be interpreted in terms of teleportation? In particular, in the teleportation protocol, Alice would have had to send *two* bits to Bob – what happened to the second bit?