Lecture & Proseminar 250078/250042
"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2022/23

— Exercise Sheet #6 —

Problem 15: LOCC protocols.
A general LOCC protocol can involve an arbitrary number of rounds of measurement and classical communication. In this problem, we will show that any LOCC protocol can be realized in a single round with only one-way communication, i.e., a protocol involving just the following steps: Alice performs a single measurement described by POVM operators M_j, sends the result j to Bob, and Bob performs a unitary operation U_j on his system.
The idea is to show that the effect of any measurement which Bob can do can be simulated by Alice – in a specific sense, namely up to local unitaries – so all of Bob's actions can be replaced by actions by Alice, except for a final unitary rotation.

1. First, suppose Alice and Bob share the state $|\psi\rangle = \sum \lambda_l |l\rangle_A |l\rangle_B$, and suppose Bob performs a measurement with POVM operators $K_j = \sum_{kl} K_{j,kl} |k\rangle_B \langle l|_B$. Let us denote the post-measurement state by $|\alpha_j\rangle$. On the other hand, suppose that Alice does a measurement with POVM operators with operators $L_j = \sum_{kl} K_{j,kl} |k\rangle_A \langle l|_A$, and denote the post-measurement state by $|\beta_j\rangle$.

   Show that there exist unitaries V_j on system A and W_j on system B such that $|\alpha_j\rangle = (V_j \otimes W_j)|\beta_j\rangle$.

2. Use this to explain how Alice can simulate any POVM measurement of Bob, and how this can be used to implement an arbitrary multi-round protocol with a single POVM measurement $\{M_j\}$ performed by Alice, followed by a unitary operation $\{U_j\}$ on Bob's side by Bob which depends on Alice's outcome.
   (*Hint:* The bases $|l\rangle_A$ and $|l\rangle_B$ above could be an arbitrary orthonormal basis!)

Problem 16: Majorization
In this problem, we prove that $x \prec y$ implies that $x = \sum_j q_j P_j y$ for some probability distribution q_j and permutation matrices P_j, where $x, y \in \mathbb{R}^d_{\geq 0}$. The proof will proceed by induction in the dimension d of the space.

1. Let $x, y \in \mathbb{R}^d_{\geq 0}$, $x \prec y$, and let the entries of x and y (denoted by x_k, y_k) be ordered descendingly.
2. Show that there exist k and $t \in [0, 1]$ such that $x_1 = t y_1 + (1 - t) y_k$. For which k does this work? For the following steps, we choose the *smallest such* k.
3. Define $D = tI + (1 - t)T$, where T is the permutation matrix which transposes the 1st and k-th matrix elements. What are the components of the vector Dy?
4. Define x' and y' by eliminating the first entry from x and Dy, respectively. Show that $x' \prec y'$.
5. Show that this way, we can inductively prove the claim.

Problem 17: Fidelity.

1. Prove that for normalized vectors $|\psi\rangle$ and $|\phi\rangle$,

$$\big|\langle\psi|O|\psi\rangle - \langle\phi|O|\phi\rangle\big| \leq \sqrt{8}\,\sqrt{1 - |\langle\psi|\phi\rangle|}\,\|O\|_\infty\ ,$$

   with $\|O\|_\infty = \|O\|_{\text{op}} = \sup_{|\psi\rangle} \frac{\|O|\psi\rangle\|}{\||\psi\rangle\|}$. Use this to prove

$$\big|\langle\psi|O|\psi\rangle - \langle\phi|O|\phi\rangle\big| \leq 2\sqrt{\delta}\|O\|_\infty\ , \qquad (*)$$

   to leading order in δ, where $\delta = 1 - F$, with $F = |\langle\psi|\phi\rangle|^2$ the fidelity.

2. Use the operator Hölder inequality

$$|\mathrm{tr}(AB)| \leq \|A\|_1 \|B\|_\infty \ ,$$

where the *trace norm* $\|A\|_1$ is the sum of the singular values of A (i.e. for hermitian A the sum of the absolute value of the eigenvalues) to prove $(*)$ directly (and without the need for a leading-order approximation).
(Of course, any alternative proof is also fine.)