1. Administrative remarks

Quantum information, computation, algorithms

András Molnár / Norbert Schuch

Lecture information:

- 6 credits, lecture 2x 90 min/week.
- Oral exam at the end of semester.

  2-3 dates at least.

- Prof. Schuch takes over around 20/11
- Lecture notes will be available at Prof. Schuch's website / Moodle

  Past lecture notes are there as well, I will mostly follow those notes.

- prof. Norbert Schuch's webpage.
- John Preskill: Quantum comp. lecture notes
- Nielsen & Chuang: Quantum info and comp.

- Requirements: no q. mechanics/physics knowledge is needed, but a good understanding of linear alg. is required. If unsure, check the past lecture notes.
  - eigenvalues/vectors (finite dim, over $\mathbb{C}$, FAST)
  - positivity, unitarity
  - tensor product (finite dim)
  - proofs!

- Exercise classes: 4 credits, 1×90 min/week.
  - 2 groups: José Garre Rubio on Wednesday
    András Molnár on Tuesday
  - Recommended to acquire suff. practice
  - Format: exercises are solved by students.
  - Requirements: presence + present $\left[\frac{\#\text{exc.}}{\#\text{students}}\right]$ solutions at the blackboard. Roughly 30 min/exercise
  - No ex. class 1st week.

- <u>Questions?</u>

2. Introduction

<u>Quantum</u> <u>information theory</u>

- "Physics of small things": electrons, atoms, molecules.
- Describes how to manipulate / transfer / store (digital) information, deal with <u>noise</u>.

Classical information theory: abstract such that one can disregard the physical device used to store the information. However, when the info storage is inherently quantum, then this abstraction does not work the same way. This semester you will learn about the theory that is valid in this case.

Word of caution: quantum is not equivalent to small. What is small anyway?

→ molecule typical size ≈ 0.1 nm.

→ transistor current size < 10 nm.

Just because transister is small, you don't need to throw away classical info. theory. Of course, understanding how they work requires quantum physics + some quantum effects are visible.

Quantum computers, computers that use / will use quantum degrees of freedom use different principles to store and manipulate information.

Same way, quantum does not necessarily mean small. For example superconductors (e.g. magnets used in MRI/ levitating trains) are quantum, and they are macroscopic. Reason: collective behaviour of electrons. Typically, it's hard to find/engineer these type of phenomena ⇝ classical physics works well to describe the majority of the world.

Why are quantum computers hard to engineer? <u>NOISE</u>: either the quantum bits = QUBITS are easy to manipulate ⇒ they are also easily effected by noise, or they are in general hard to manipulate.

Plan:

- Linear algebra refresh
  - Positivity
  - Tensor product
- Quantum mechanics "axioms"
  - Discrete systems ↔ classially: "coin"
  - Mixed states ↔ classically: probability theory
  - Pure states ↔ classically: specifying wheather a coin is head / tail
  - Measurement and time evolution
- Entanglement: interplay between tensor product and positivity.
  - Bell inequalities
  - Teleportation, dense coding

Norbert :

- Quantum computing and algorithms
- Quantum error correction

3. Linear algebra

Finite dimensional Hilbert space over $\mathbb{C}$:

- vector space $\mathcal{H} \simeq \mathbb{C}^d$

- $\langle \cdot | \cdot \rangle$ scalar product: $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$
  - Conj. symmetric: $\forall a, b \in \mathcal{H}$

    $\langle a | b \rangle = \overline{\langle b | a \rangle}$
  - Sesquilinear: $\forall a, b, c \in \mathcal{H}$

    $\langle \lambda a + \mu b | c \rangle = \bar{\lambda} \langle a | c \rangle + \bar{\mu} \langle b | c \rangle$

    $\langle a | \lambda b + \mu c \rangle = \lambda \langle a | b \rangle + \mu \langle a | c \rangle$

  ! First variable is conj. linear !
  - Positive definite:

    $\langle a | a \rangle \geqslant 0$, $\langle a | a \rangle = 0 \iff a = 0$.

Notations:

- usually we write $| v \rangle \in \mathcal{H}$,

  $| \cdot \rangle$ is called "ket" (from bracket, as it is half or $\langle \cdot | \cdot \rangle$)

- Similarly, we write $\langle v|$ for the linear functional def. by $w \mapsto \langle v|w\rangle$.

$\langle \cdot |$ is called "bra" (as it is the first part of the bra-cket).

Usually, we implicitly fix an orthonormal basis. This basis is called the "computational basis", as all computations are going to be carried out in matrix form. Elements of this basis are denoted by

$|0\rangle, |1\rangle, \dots, |d-1\rangle$.

Then a general vector is

$$|v\rangle = v_0|0\rangle + v_1|1\rangle + \dots + v_{d-1}|d-1\rangle =$$

$$= \sum_{i=0}^{d-1} v_i |i\rangle .$$

Matrix notation:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Big\} d, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, |d\rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Then

$$|v\rangle = \sum_{i=0}^{d-1} v_i |i\rangle = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{d-1} \end{pmatrix}.$$

! Column vector !

Orthonormality of the basis:

$$\langle i | j \rangle = \delta_{ij} \quad \left(\text{Kronecker delta: } \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}\right)$$

This means that for $|v\rangle, |w\rangle \in \mathcal{H}$,

if $|v\rangle = \sum_i v_i |i\rangle$ and $|w\rangle = \sum_i w_i |i\rangle$, then

$$\langle v | w \rangle = \sum_{ij} \bar{v}_i w_j \langle i | j \rangle = \sum_i \bar{v}_i w_i .$$

We can thus write

$$\langle v| = \sum_i \bar{v}_i \langle i| = (|v\rangle)^{+} = (\bar{v}_0, \ldots, \bar{v}_{d-1})$$

Indeed,

$$(\bar{v}_0, \ldots, \bar{v}_{d-1}) \begin{pmatrix} w_0 \\ \vdots \\ w_{d-1} \end{pmatrix} = \sum_{i=0}^{d-1} \bar{v}_i w_i .$$

We write

$$\| v \| = \sqrt{\langle v | v \rangle} = \left(\sum_{i=0}^{d-1} |v_i|^2 \right)^{1/2}$$

↳ remember : ≥ 0.

We call $\|a\|$ the <u>norm</u> of a.

<u>Remark</u>: sometimes people also think about p-norms:

$$\| v \|_p = \left(\sum_i |v_i|^p \right)^{1/p}.$$

In finite dimensions $\|v\|_p < \infty \ \forall |v\rangle \in \mathcal{H}$.

Then $\|a\| \equiv \|a\|_2$.

<u>Linear maps</u>:

$M: \mathcal{H}_1 \to \mathcal{H}_2$ is linear if $\forall |v\rangle, |w\rangle \in \mathcal{H}_1$, $\lambda, \mu \in \mathbb{C}$

$$M(\lambda |v\rangle + \mu |w\rangle) = \lambda M|v\rangle + \mu M|w\rangle.$$

M is completely determined by its action on the basis vectors:

$$M|v\rangle = \sum_j v_j \cdot M|j\rangle \quad \text{if } |v\rangle = \sum_{i=0}^{d-1} v_i |i\rangle.$$

We can also decompose $M|j\rangle$ in the basis:

$$M|j\rangle = \sum_i M_{ij} |i\rangle.$$

Then

$$M|v\rangle = \sum_j v_j M|j\rangle = \sum_{ij} M_{ij} \cdot v_j \; |i\rangle.$$

Matrix notation: $d_1 = \dim(\mathcal{H}_1)$, $d_2 = \dim(\mathcal{H}_2)$

$$M = \begin{pmatrix} M_{00} & M_{01} & \dots & M_{0\,d_1-1} \\ M_{10} & & & \\ \vdots & & & \\ M_{d_2-1,0} & & & M_{d_2-1,\,d_1-1} \end{pmatrix}$$

Then:

$$M|v\rangle = \begin{pmatrix} M_{00} & \dots & M_{0\,d_1-1} \\ \vdots & & \\ M_{d_2-1\,0} & \dots & M_{d_2-1\,d_1-1} \end{pmatrix} \begin{pmatrix} v_0 \\ \vdots \\ v_{d_1-1} \end{pmatrix} =$$

$$= \begin{pmatrix} M_{00} \cdot v_0 + \dots + M_{0d-1} \cdot v_{d-1} \\ \vdots \\ M_{d-1\,0} \cdot v_0 + \dots + M_{d-1\,d-1}\, v_{d-1} \end{pmatrix}.$$

Let us notice that

$$\langle i|M|j\rangle = \langle i|\left(\sum_k M_{kj}\,|k\rangle\right)$$

$$= \sum_k M_{kj}\,\langle i|k\rangle = M_{ij}.$$

We can also write

$$M = \sum_{ij} M_{ij}\,|i\rangle\langle j|\,,$$

as

$$\sum_{ij} M_{ij}\,|i\rangle\underbrace{\langle j|k\rangle}_{\delta_{jk}} = \sum_i M_{ik}|i\rangle = M|k\rangle.$$

Here:

$$|i\rangle\langle j| = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \leftarrow i$$

$$\uparrow j$$

Important matrices:

$$X = \sigma_x = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |1\rangle\langle 0| + |0\rangle\langle 1|$$

$$Y = \sigma_y = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i|1\rangle\langle 0| - i|0\rangle\langle 1|$$

$$Z = \sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$I = \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |0\rangle\langle 0| + |1\rangle\langle 1|$$

$$\sigma_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = |1\rangle\langle 0| = X + iY$$

$$\sigma_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 1| = X - iY$$

Remark: we work w/ column vectors, and thus we define product as

$$(AB)|v\rangle := A(B|v\rangle),$$

i.e.

AB means "B acts first, A second".

This way the product is compatible w/ matrix product:

$$A(B|v\rangle) = A\left(\sum_{jk} B_{jk} v_k \cdot |j\rangle\right) =$$

$$= \sum_{jk} B_{jk} v_k \cdot A|j\rangle =$$

$$= \sum_{ijk} \underbrace{A_{ij} B_{jk}}_{\text{usual matrix product}} v_k \, |i\rangle .$$

For example:

$$XY = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = iZ.$$

$$YX = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -iZ$$

Same way:

$$YZ = iX = -ZY$$

$$ZX = iY = -XZ$$

Also:

$$X^2 = Y^2 = Z^2 = \mathbb{1}.$$

Shorthand:

$$\sigma_j \sigma_k = i \underline{\varepsilon_{jkl}} \sigma_l + \delta_{jk} \cdot \mathbb{1}.$$

↑ completely antisymmetric tensor:

$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$

$\varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1$

$\forall$ other entry is 0.

<u>Adjoint</u>: let $M: \mathcal{H}_1 \to \mathcal{H}_2$ be lin. map, then there is a unique map, $M^\dagger$, s.t.

$$\langle v | M w \rangle = \langle M^\dagger v | w \rangle = \overline{\langle w | M^\dagger v \rangle}$$

(Proof: straightforward).

Explicit construction:

$$(M^\dagger)_{ij} = \overline{M_{ji}},$$

the matrix is conjugate traspose.

<u>Indeed</u>:

$$\overline{(M^\dagger)_{ij}} = \overline{\langle i | M^\dagger j \rangle} = \langle M^\dagger j | i \rangle =$$

$$= \langle j | M | i \rangle = M_{ji}.$$

In matrix form:

$$\begin{pmatrix} M_{00} & \cdots & M_{0d-1} \\ \vdots & & \vdots \\ M_{d-10} & & M_{d-1d-1} \end{pmatrix}^\dagger = \begin{pmatrix} \overline{M_{00}} & \cdots & \overline{M_{d-10}} \\ \vdots & & \\ \overline{M_{0d-1}} & & \overline{M_{d-1d-1}} \end{pmatrix}$$

Def We say that $M: \mathcal{H} \to \mathcal{H}$ is Hermitian/self-adjoint if $M^{+} = M$.

Def We say that U is unitary if

$$UU^{+} = U^{+}U = \mathbb{1}.$$

Remark: in finite dimensions both $UU^{+} = 1$ and $U^{+}U = \mathbb{1}$ implies already that U is unitary. (U is a square matrix)

<u>Example</u>:

$$X^{+} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{+} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X$$

$$Y^{+} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^{+} = \begin{pmatrix} 0 & \bar{i} \\ \overline{-i} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = Y$$

$$Z^{+} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{+} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z.$$

As

$$X^2 = Y^2 = Z^2 = \mathbb{1}$$

and they are self-adjoint, they are unitaries:

$$XX^{+} = YY^{+} = ZZ^{+} = \mathbb{1}.$$

Norm of linear maps:

Let $M: \mathcal{H}_1 \to \mathcal{H}_2$ be linear.

Then the norm of M is

$$\|M\| = \sup_{\substack{v \in \mathcal{H}_1 \\ v \neq 0}} \frac{\|Mv\|}{\|v\|} .$$

Eigenvalues, eigenvectors

Let $M \in \mathrm{End}(\mathcal{H})$. If

$M|v\rangle = \lambda |v\rangle$ for some $\lambda \in \mathbb{C}$, $|v\rangle \in \mathcal{H}$, $|v\rangle \neq 0$,

Then

- v is called an eigen-vector of M.
- λ is called an eigenvalue of M.

The set of eigenvalues is called the spectrum of M.

How to find them:

$|v\rangle \in \mathrm{Ker}(M - \lambda \mathbb{1})$

So solutions of

$\det(M - \lambda \mathbb{1}) = 0$ are the spectrum.

Eig. vect. corr. to diff. eig. vecter are lin. indep.

# of eig. vecters $\leq$ dim $(\mathcal{H})$.

(=) : Diagonalizable, $M = X^{-1} D X$ w/ D diagonal

(<) : Not diagonalizable.

Eg.

$\sigma_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ has a unique eig. value : 0, and for it, unique eig. vecter $|1\rangle$.

Sufficient condition for diagonalizability:

$[M, M^\dagger] = 0.$

For example, all Hermitian and all unitary matrices are diagonalizable.

Hermitian matrices are diagonalizable by a unitary:

$$M = \mathcal{U} D \mathcal{U}^\dagger = \mathcal{U} \sum_k \lambda_k |k\rangle\langle k| \mathcal{U}^\dagger =$$

$$= \sum_k \lambda_k |v_k\rangle\langle v_k|$$

$$= \sum_{\lambda \in Spec(M)} \lambda P_\lambda = \sum_{\lambda \in Spec(M)} \lambda \sum_k |v_{k,\lambda}\rangle\langle v_{k,\lambda}|$$

Eg:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{Char. poly: } \lambda^2 - 1 = 0 \rightarrow \text{Spec} = \{+1, -1\}$$

Eig. vectors: $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$ for $+1$.

$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$ for -1.

$$X = |+\rangle\langle+| - |-\rangle\langle-| = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}(1\ 1) -$$

$$-\frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}(1\ -1) =$$

$$= \frac{1}{2}\left[\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Y, Z : similar, same spectrum!

☆ End of 1^{st} lecture.

Recap:

- Complex, finite dim. vector spaces w/ scalar product (Hilbert spaces)

- Bra-ket notation:

  - vectors: $|v\rangle$ "ket" $\begin{pmatrix} v \in \text{Hor} \\ v \in \mathbb{N} \end{pmatrix}$

  - "comp." basis: $|0\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, |d-1\rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

- The inner product is

$$\langle w|v\rangle = \sum_{i=0}^{d-1} \bar{w}_i v_i = (\bar{w}_0 \ldots \bar{w}_{d-1}) \begin{pmatrix} v_1 \\ \vdots \\ v_{d-1} \end{pmatrix}$$

- I write $\langle w|$ for the lin. fcn. def. by $|v\rangle \mapsto \langle w|v\rangle$,
  $\langle w| = (\bar{w}_0 \ldots \bar{w}_{d-1})$. "bra"

- Lin. ops = matrices (in comp. basis)

$$A = \sum_{ij} A_{ij} |i\rangle\langle j| = \begin{pmatrix} A_{00} & \ldots & A_{0d-1} \\ \vdots & & \vdots \\ A_{d-10} & \ldots & A_{d-1d-1} \end{pmatrix}$$

- Adjoint:

  - $\langle v|Aw\rangle = \langle A^+ v|w\rangle$

  - $$A^+ = \sum_{ij} \bar{A}_{ij} |j\rangle\langle i| \doteq \begin{pmatrix} \bar{A}_{00} & \ldots & \bar{A}_{d-10} \\ \vdots & & \vdots \\ \bar{A}_{0d-1} & \ldots & \bar{A}_{d-1d-1} \end{pmatrix}$$

- We have defined Hermitian (= self-adjoint) and unitary matrices:
  - A is Hermitian if $A = A^{\dagger}$.
  - U is unitary if it is invertible and $U^{\dagger} = U^{-1}$.
- Eigenvalues/vectors $\lambda \in \mathbb{C}$, $|v\rangle \in \mathcal{H}$, $|v\rangle \neq 0$:

  $$A|v\rangle = \lambda |v\rangle$$
- For Hermitian/unitary matrix M $\exists U$ unitary s.t.

  $$M = U D U^{\dagger}, \quad D = \sum_i D_i |i\rangle\langle i| \text{ diagonal.}$$

  __Note:__ eig. values are D_i, eig. vectors are $U|i\rangle$.

__Today:__
- Positivity
- Tensor product

Def: We say that $M: \mathcal{H} \to \mathcal{H}$ is positive semidefinite if $\forall |v\rangle \in \mathcal{H}$

$$\langle v| M |v\rangle \geq 0.$$

We write $M \geq 0$ if M is positive semidefinite.

We say that M is positive definite if $\forall |v\rangle \in \mathcal{H}$, $|v\rangle \neq 0$

$$\langle v| M |v\rangle > 0.$$

We write $M > 0$ if M is positive definite.

I will say positive instead of pos. semidefinite.

For example,

- $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is positive (def.), as

$$\langle v|v\rangle \geq 0 \text{ and } \langle v|v\rangle = 0 \Rightarrow |v\rangle = 0.$$

- $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is not positive as

$$\langle 1|Z|1\rangle = (0\ 1)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1 < 0.$$

- Diag. mx ≥ 0 iff $\forall$ entry ≥ 0.

Thm: Let $M: \mathcal{H} \to \mathcal{H}$ be linear. The following are equivalent:

(1) M is positive: $\langle v|M|v\rangle \geq 0 \quad \forall v \in \mathcal{H}$

(2) M is Hermitian and all of its eig. values are ≥ 0

(3) $\exists X: \mathcal{H} \to \mathcal{H}$ s.t. $M = X^{\dagger} X$.

Proof: (1) $\Rightarrow$ (2) Let $|v\rangle, |w\rangle \in \mathcal{H}$, then

$$\langle v+w|M|v+w\rangle - \langle v|M|v\rangle - \langle w|M|w\rangle =$$

$$= \langle v|M|w\rangle + \langle w|M|v\rangle \in \mathbb{R}$$

$$\langle v+iw|M|v+iw\rangle - \langle v|M|v\rangle - \langle w|M|w\rangle =$$

$$= i\langle v|M|w\rangle - i\langle w|M|v\rangle \in \mathbb{R},$$

or

$$\langle v|M|w\rangle - \langle w|M|v\rangle \in i\mathbb{R}.$$

This means that

$$\langle w|M|v\rangle = \overline{\langle v|M|w\rangle} = \overline{\langle M^{\dagger}v|w\rangle} = \langle w|M^{\dagger}|v\rangle.$$

As it holds $\forall\, |v\rangle, |w\rangle \in \mathcal{H}$, $M = M^{\dagger}$, i.e. M is self-adjoint.

Let now $|v\rangle$ be an eig. vector w/ eig. value λ, then

$$\langle v|Mv\rangle = \langle v|\lambda v\rangle = \lambda\langle v|v\rangle,$$

and thus, as $\langle v|v\rangle > 0$, $\lambda > 0$ as well.

(2) ⇒ (3) We can write

$M = \mathcal{U} D \mathcal{U}^\dagger$ w/ D diag, $\mathcal{U}$ unitary.

$D \geq 0 \Leftrightarrow \forall i \; D_i > 0$. We can then define $D^{1/2} = \sum_i \sqrt{D_i}\; |i\rangle\langle i| \geq 0$.

$M = \mathcal{U} D^{1/2} D^{1/2} \mathcal{U}^\dagger = X^\dagger X$ w/ $X = D^{1/2}\mathcal{U}^\dagger$.

(3) ⇒ (1)

$$\langle v|X^\dagger X|v\rangle = \langle Xv|Xv\rangle \geq 0.$$

The set of self-adjoint matrices form a <u>real</u> vector space:

- $A = A^+$, $B = B^+ \Rightarrow (A+B)^+ = A^+ + B^+ = A + B$
- $A = A^+$, $\lambda \in \mathbb{R} \Rightarrow (\lambda A)^+ = \lambda A^+ = \lambda A$,

But for example

$$(iA)^+ = \bar{i} A^+ = -i A^+ = -iA \neq iA.$$

On this set there is a partial order:

$$A \leq B \iff 0 \leq B - A.$$

<u>Exercise #1</u> show that if $A \leq B$, then

- $XAX^+ \leq XBX^+$ for any (even non-square) X.

<u>Exercise #2</u>: find example s.t.

$$A \leq B, \text{ but } CA \not\leq CB.$$

Tensor product of vector spaces:

Def: Let $\mathcal{H}_A, \mathcal{H}_B$ be (finite dim.) Hilbert spaces, with ONB $\{|i\rangle_A\}_{i=0}^{d_A-1}$ and $\{|j\rangle_B\}_{j=0}^{d_B-1}$.

Their tensor product is another Hilb. space with dim. $d_A \cdot d_B$ and ONB

$$\left\{ |i\rangle_A \otimes |j\rangle_B \,\middle|\, i = 0..d_A-1,\ j = 0..d_B-1 \right\}.$$

I will also write $|ij\rangle$ for $|i\rangle \otimes |j\rangle$.

Any vector from $\mathcal{H}_A \otimes \mathcal{H}_B$ is of the form:

$$|v\rangle = \sum_{ij} v_{ij} \, |ij\rangle$$

Scalar product of two generic vectors in $\mathcal{H}_A \otimes \mathcal{H}_B$:

$$\langle v|w\rangle = \sum_{ijkl} \overline{v_{ij}} \, w_{kl} \, \langle ij|kl\rangle$$

$$= \sum_{ijkl} \overline{v_{ij}} \, w_{kl} \, \delta_{ik} \delta_{jl} =$$

$$= \sum_{ij} \overline{v_{ij}} \, w_{ij} .$$

The tensor product space has the following "product" property:

$\forall v \in \mathcal{H}_A,\ w \in \mathcal{H}_B$ there is a vector $|v\rangle \otimes |w\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ s.t. if

$$|v\rangle = \sum_i v_i |i\rangle, \quad |w\rangle = \sum_j w_j |j\rangle,$$

then

$$|v\rangle \otimes |w\rangle = \sum_{ij} v_i w_j |ij\rangle$$

Notice that:

- $|\lambda_1 v_1 + \lambda_2 v_2\rangle \otimes |w\rangle = \lambda_1 |v_1\rangle \otimes |w\rangle + \lambda_2 |v_2\rangle \otimes |w\rangle$
- $|v\rangle \otimes |\mu_1 w_1 + \mu_2 w_2\rangle = \mu_1 |v\rangle \otimes |w_1\rangle + \mu_2 |v\rangle \otimes |w_2\rangle$
- $|\lambda v\rangle \otimes |w\rangle = |v\rangle \otimes |\lambda w\rangle = \lambda \cdot |v\rangle \otimes |w\rangle$.

In fact, one can define tensor product basis-independently; these are the defining relations.

Example: $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2$. Then $\mathcal{H}_A \otimes \mathcal{H}_B$ is 4 dimensional, an ONB is

$|00\rangle, |01\rangle, |10\rangle, |11\rangle$.

That is, $\forall$ vector is of the form

$$|v\rangle = v_{00}|00\rangle + v_{01}|01\rangle + v_{10}|10\rangle + v_{11}|11\rangle.$$

Note: we have already seen tensor product: the set of $\mathcal{H} \to \mathcal{H}$ linear (I will write $B(\mathcal{H})$ for this set) is $B(\mathcal{H}) \cong \mathcal{H} \otimes \mathcal{H}^*$, where $\mathcal{H}^*$ is the set of lin. fcn. on $\mathcal{H}$:

- any lin. func. is of the form
  $|v\rangle \mapsto \langle w|v\rangle$ for some $|w\rangle \in \mathcal{H}$.
- So a basis of $\mathcal{H}^*$ is $\{\langle i|\}_{i=0}^{d-1}$.
- Therefore elements of $\mathcal{H} \otimes \mathcal{H}^*$ are
  $\sum_{ij} M_{ij} |i\rangle\langle j|$, that is, matrices.

Careful!

It is tempting to think that every element of $\mathcal{H}_A \otimes \mathcal{H}_B$ can be written as

$$|v\rangle \otimes |w\rangle = \sum_{ij} v_i w_j \, |ij\rangle,$$

but it is not true.

For example: elements of $\mathcal{H} \otimes \mathcal{H}^*$ are matrices. Elements of $\mathcal{H} \otimes \mathcal{H}^*$ that are tensor product are of the form

$$|v\rangle\langle w| = \sum_{ij} v_i w_j \, |i\rangle\langle j| \quad \text{! notation !}$$

These are all rank-1 matrices,

If $M = |v\rangle\langle w|$, then

$$M|x\rangle = |v\rangle\underbrace{\langle w|x\rangle}_{c \in \mathbb{C}} = c \cdot |v\rangle .$$

There are not rank-1 operators, e.g. $\mathbb{1}$.

So again, not all elements of $\mathcal{H}_A \otimes \mathcal{H}_B$ is of the form $|v\rangle \otimes |w\rangle \hat{=} |v\rangle|w\rangle$.

<u>Def</u>: Schmidt rank: Let $\mathcal{H}_A, \mathcal{H}_B$ be Hilbert spaces, $|v\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.
The <u>Schmidt rank</u> of $|v\rangle$ is the smallest $r \in \mathbb{N}$ s.t. one can write

$$|v\rangle = \sum_{i=1}^{r} |u_i\rangle \otimes |w_i\rangle .$$

Such a decomp. is called min. rank decomposition.
If $\mathcal{H}_B = \mathcal{H}_A^*$, this is just the rank of the matrix:

$$M = \sum_{i=1}^{r} |u_r\rangle\langle w_r| .$$

If $\mathcal{H}_B \neq \mathcal{H}_A^*$: Still can write

$$|v\rangle = \sum_{ij} v_{ij} \, |i\rangle \otimes |j\rangle$$

And we can form the matrix

$$\begin{pmatrix} v_{0,0} & \cdots & v_{0,d_B-1} \\ \vdots & & \vdots \\ v_{d_A-1,0} & & v_{d_A-1,d_B-1} \end{pmatrix}$$

Then the rank of this matrix is the Schmidt rank of $|v\rangle$:

If $v_{ij} = \sum_{k=1}^{r} u_{ik} w_{kj}$, then

$$\sum_{ij} v_{ij} \, |i\rangle \otimes |j\rangle = \sum_{k=1}^{r} \sum_i u_{ik} |i\rangle \otimes \sum_j w_{kj} |j\rangle$$

$$= \sum_{k=1}^{r} |u_k\rangle \otimes |w_k\rangle$$

And vice verse.

Notice: the vectors $\{|u_k\rangle\}$ and $\{|w_k\rangle\}$ are lin. independent. ☆ End of 2nd

23/10/09 : Tensor product, quant. mechanics

Ex. class?
Math-phys: OK

Recap : (Reminder: ex. class separate!)

Hilbert space: finite dim. complex v-space w/ scalar product.

Bra-ket notation:

$|v\rangle \in \mathcal{H}$ "ket" $\rightarrow \begin{cases} v \in \mathcal{H} \\ v = 0, 1, \ldots \end{cases}$

$\langle w| \in \mathcal{H}^*$ "bra" $w \in \mathcal{H}$, $|v\rangle \mapsto \langle w|v\rangle$

$$\langle \lambda_1 w_1 + \lambda_2 w_2| = \bar{\lambda}_1 \langle w_1| + \bar{\lambda}_2 \langle w_2|$$

ONB: $|v\rangle = \sum_i v_i |i\rangle = \begin{pmatrix} v_0 \\ \vdots \\ v_{d-1} \end{pmatrix}$

$$\langle w| = \sum_i \bar{w}_i \langle i| = (\bar{w}_0 \ldots \bar{w}_{d-1})$$

Scalar prod.

$$\langle w|v\rangle = \sum_i \bar{w}_i w_i = (\bar{w}_0 \ldots \bar{w}_{d-1}) \begin{pmatrix} v_0 \\ \vdots \\ v_{d-1} \end{pmatrix}.$$

→ Lin. ops → matrices $A = \sum A_{ij} |i\rangle\langle j|$

→ Adjoint → conj. transpose in ONB ($A \mapsto A^+$)

→ Self-adjoint: $A = A^+$ ($A \in \mathcal{B}(\mathcal{H})$)

→ Unitary : $\mathcal{U}^+ = \mathcal{U}^{-1}$ ($\mathcal{U} \in \mathcal{B}(\mathcal{H})$)

→ If M is unitary / self adjoint, then $M = \mathcal{U} D \mathcal{U}^+$ w/ $\mathcal{U}$ unitary, D diag.

→ Positivity: $A \in \mathcal{B}(\mathcal{H})$ $\quad A \geq 0$ iff

(1) $\forall |v\rangle \in \mathcal{H} \qquad \langle v | A v \rangle \geq 0$

(2) $A = A^\dagger$ & eig. values ≥ 0

(3) $A = X X^\dagger$

→ Tensor product:

$$\mathcal{H}_A \otimes \mathcal{H}_B = \text{Span}\{ |i\rangle \otimes |j\rangle \mid \{|i\rangle\}_{i=0}^{d_A - 1} \text{ ONB of } \mathcal{H}_A, \ \{|j\rangle\}_{j=0}^{d_B-1} \text{ ONB of } \mathcal{H}_B \}$$

Tensor prod. of vectors:

$$|v\rangle \otimes |w\rangle = \sum_{ij} v_i w_j \, |i\rangle \otimes |j\rangle = \sum_{ij} v_i w_j \, |ij\rangle$$

$$\mathcal{H}_A \otimes \mathcal{H}_B = \text{Span}\{ |v\rangle \otimes |w\rangle \mid |v\rangle \in \mathcal{H}_A, |w\rangle \in \mathcal{H}_B \}$$

$$\neq \{ |v\rangle \otimes |w\rangle \mid v \in \mathcal{H}_A, |w\rangle \in \mathcal{H}_B \}$$

Note: $\mathcal{H}_A \otimes \mathcal{H}_B^* \cong \text{Lin}(\mathcal{H}_B, \mathcal{H}_A)$. Indeed, elements of it are $\sum_{ij} M_{ij} |i\rangle\langle j|$,

so $M \in \mathcal{H}_A \otimes \mathcal{H}_B^*$ acts on $\mathcal{H}_B$ as

$$|v\rangle \mapsto \sum_{ij} M_{ij} \, |i\rangle \underbrace{\langle j | v\rangle}_{\mathbb{C}} \in \mathcal{H}_A .$$

We have seen: Schmidt rank: min. r s.t.

$$|v\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \, ; \ |v\rangle = \sum_{i=1}^{r} |u_i\rangle \otimes |w_i\rangle$$

For lin. ops: Schmidt rank = matrix rank.

In fact, also true for not lin. ops.

$|v\rangle = \sum_{ij} v_{ij} |ij\rangle \rightarrow$ Schmidt rank is the rank of the matrix v_{ij}.

<u>Today</u>:
- Schmidt decomp / SVD
- Tens. prod. of mult. spaces
- Tens. pord. of matrices

<u>Thm</u>: Let $|v\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ w/ Schmidt rank r. Assume

$$|v\rangle = \sum_{i=1}^{S} |u_i\rangle \otimes |w_i\rangle$$

Then $r = S \iff \{u_i\}$ are lin. indep. and $\{w_i\}$ are lin. indep.

<u>Proof</u>: $\boxed{\Rightarrow}$ Assume $\{u_i\}$ are not lin. indep.

Then w.l.o.g. $u_S = \sum_{i=1}^{S-1} \lambda_i u_i$, so

$$|v\rangle = \sum_{i=1}^{S-1} |u_i\rangle \otimes |w_i\rangle + \sum_{i=1}^{S-1} \lambda_i |u_i\rangle \otimes |w_S\rangle$$

$$= \sum_{i=1}^{S-1} |u_i\rangle \otimes |w_i + \lambda_i w_S\rangle \Rightarrow S > r.$$

$\boxed{\Leftarrow}$ Let $|v\rangle = \sum_{i=1}^{s} |u_i\rangle \otimes |w_i\rangle = \sum_{i=1}^{r} |\hat{u}_i\rangle \otimes |\hat{w}_i\rangle$

with $\{w_i\}$ lin. independent.

Then $\exists \langle x_i|$ s.t. $\langle x_i | w_j\rangle = \delta_{ij}$

$$|u_i\rangle = \sum_{j=1}^{s} |u_j\rangle \cdot \langle x_i | w_j\rangle =$$

$$= \sum_{j=1}^{r} |\hat{u}_j\rangle \cdot \langle x_i | \hat{w}_j\rangle = \sum_{j=1} \lambda_j |\hat{u}_j\rangle$$

$\Rightarrow$ There is at most r lin. indep. of $\{u_i\}$. $\Rightarrow$ $s \leq r$, and as r is min, $s = r$.

$\square$

Let us consider now $|v\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$

$$|v\rangle = \sum_{i=1}^{r} |u_i\rangle \otimes |w_i\rangle$$

If r is the Shmidt rank, then such a decomp. is called a min. rank decomp. of $|v\rangle$.

Thm: Let $|v\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ w/ Schmidt rank r. Assume

$$|v\rangle = \sum_{i=1}^{r} |u_i\rangle \otimes |w_i\rangle = \sum_{j=1} |x_j\rangle \otimes |y_j\rangle$$

Then $\exists A$ $r\times r$ invertible matrix s.t.

$$|u_i\rangle = \sum_j A_{ij} |x_j\rangle,$$

$$|w_i\rangle = \sum_j (A^{-1})_{ji} |y_j\rangle$$

Proof: As $|w_i\rangle$ are lin. indep, $\exists |z_j\rangle$ s.t.

$\langle z_j | w_i\rangle = \delta_{ij}$. Then

$$|u_j\rangle = \sum_{i=1}^{r} |u_i\rangle \langle z_j | w_i\rangle =$$

$$= \sum_{i=1}^{r} |x_i\rangle \langle z_j | y_i\rangle = \sum_{i=1}^{r} A_{ji} |x_i\rangle$$

As $|u_j\rangle$ are lin. indep., the $r\times r$ matrix A_{ji} is invertible.

$$\sum_{i=1}^{r} |u_i\rangle \otimes |w_i\rangle = \sum_{i=1}^{r} \sum_{j=1}^{r} A_{ij} |x_j\rangle \otimes |w_i\rangle$$

$$= \sum_{j=1}^{r} |x_j\rangle \otimes \Big| \sum_{i=1}^{r} A_{ij} w_i\Big\rangle = \sum_{j=1}^{r} |x_j\rangle \otimes |y_j\rangle$$

So $|y_j\rangle = \sum_i A_{ij} |w_i\rangle$, or

$$|w_i\rangle = \sum_k (A^{-1})_{ki} |y_k\rangle .$$

$\square$

<u>Thm</u> (Schmidt decomposition): $|v\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ w/ Schmidt rank r. Then there is a min. rank decomp.

$$|v\rangle = \sum_i \lambda_i |c_i\rangle \otimes |d_i\rangle \quad \text{w/}$$

$$\langle c_i | c_j \rangle = \delta_{ij} , \quad \langle d_i | d_j \rangle = \delta_{ij} , \quad \lambda_i > 0 .$$

<u>Proof</u>: using the pevious thm, we can orthogonalize one side:

$$|v\rangle = \sum_{i=1}^{r} |a_i\rangle \otimes |b_i\rangle \text{ s.t. } \langle a_i | a_j \rangle = \delta_{ij} .$$

Then the matrix M def. by

$$M_{ij} = \langle b_i | b_j \rangle$$

might have writen $\langle v | M | v \rangle$

is positive:

$$\sum_{ij} \bar{\alpha}_i M_{ij} \alpha_j = \sum_{ij} \bar{\alpha}_i \alpha_j \langle b_i | b_j \rangle = \langle \sum_i \alpha_i b_i | \sum_j \alpha_j b_j \rangle \geq 0,$$

so diagonalizable by a unitary: $M = UDU^\dagger$, Or

$$M_{ij} = \sum_{kl} u_{ik} \cdot \delta_{kl} D_l \cdot \bar{u}_{jl} \quad , D_l > 0.$$

Then we can write

$$|d_k\rangle = \frac{1}{\sqrt{D_k}} \sum_i u_{ik} |b_i\rangle$$

$$|c_k\rangle = \sum_i \bar{u}_{ik} |a_i\rangle = \sum_i (u^{-1})_{ki} |a_i\rangle$$

Such that

$$|v\rangle = \sum_k D_k \, |c_k\rangle \otimes |d_k\rangle .$$

Here

- $$\langle c_k | c_l \rangle = \sum_{ij} u_{ik} \bar{u}_{jl} \langle a_i | a_j \rangle$$

$$= \sum_i u_{ik} \bar{u}_{il} = \sum_i (u^{+})_{li} u_{ik} = \delta_{kl}$$

- $$\langle d_k | d_l \rangle = \frac{1}{\sqrt{D_k}\sqrt{D_l}} \sum_{ij} \bar{u}_{ik} u_{jl} \langle b_i | b_j \rangle$$

$$= \frac{1}{\sqrt{D_k D_l}} \sum_{ij} (u^{+})_{ki} M_{ij} u_{jl} = \delta_{kl}.$$

Note: for $\mathcal{H}\otimes\mathcal{H}^* \cong B(\mathcal{H})$ the Schmidt decomposition is called Singular value decomposition (SVD):

$$M = \sum_{ij} M_{ij} |i\rangle\langle j| = \sum_{i=0}^{r-1} \lambda_i |v_i\rangle\langle w_i|$$

with $\langle v_i|v_j\rangle = \delta_{ij}$ $\langle w_i|w_j\rangle = \delta_{ij}$

Note:

$$M = \begin{pmatrix} v_0 | v_1 | \dots | v_{r-1} \end{pmatrix} \begin{pmatrix} \lambda_0 & & \\ & \ddots & \\ & & \lambda_{r-1} \end{pmatrix} \begin{pmatrix} w_0 \\ \vdots \\ w_{r-1} \end{pmatrix}$$

$$= V \Lambda W,$$

there V and W are unitaries/isometries.

Note: a way to obtain V, Λ, W is:

$$MM^+ = V\Lambda W W^+ \Lambda V^+ = V\Lambda^2 V^+$$

$$M^+M = W^+\Lambda V^+ V \Lambda W = W^+\Lambda^2 W, \text{ i.e.}$$

V diag. MM^+, W diag. M^+M; they have same spectrum.

<u>Tensor product of multiple vector spaces:</u>

$\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_k$ are Hilbert spaces w/

basis $\{|i\rangle\}_{i=0}^{d_1-1}, \dots \quad \{|i\rangle\}_{i=0}^{d_k-1}$

Then

$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_k$ is a $d_1 \cdots d_k$-dim. space

w/ ONB

$$\{ |i_1\rangle \otimes \dots \otimes |i_k\rangle \mid i_1 = 0, \dots, d_1-1, \dots, i_k = 0, \dots d_{k-1} \}$$

For the basis vectors I will also write

$$|i_1\rangle \otimes \dots \otimes |i_k\rangle = |i_1 \dots i_k\rangle$$

Elements of the space:

$$\sum_{i_1 \dots i_k} v_{i_1 \dots i_k} |i_1 \dots i_k\rangle$$

Scalar product:

$$\langle v|w\rangle = \sum_{\substack{i_1 \dots i_k \\ j_1 \dots j_k}} \overline{v}_{i_1 \dots i_k} w_{j_1 \dots j_k} \langle i_1 \dots i_k | j_1 \dots j_k\rangle$$

$$= \sum_{i_1 \dots i_k} \overline{v}_{i_1 \dots i_k} w_{i_1 \dots i_k} .$$

For example: $\mathcal{H}_1 = \mathcal{H}_2 = \dots = \mathcal{H}_n = \mathbb{C}^2$.

Then $\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_n$ is 2^n dimensional

$|v\rangle \in \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_n$

$$|v\rangle = \sum_{i_1 \dots i_n \in \{0,1\}} v_{i_1 \dots i_n} |i_1 \dots i_n\rangle$$

($v_{i_1 \dots i_n} \in \mathbb{C}$, $i_1 \dots i_n \in \{0,1\}$)

<u>Note:</u> $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \cong (\mathcal{H}_A \otimes \mathcal{H}_B) \otimes \mathcal{H}_C \cong \mathcal{H}_A \otimes (\mathcal{H}_B \otimes \mathcal{H}_C)$

<u>Also:</u> $\mathcal{H}_A \otimes \mathcal{H}_B \cong \mathcal{H}_B \otimes \mathcal{H}_A$

$$|i\rangle_A \otimes |j_B\rangle \longmapsto |j\rangle_B \otimes |i\rangle_A$$

Also for multiple vector spaces:

$$\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \cong \mathcal{H}_C \otimes \mathcal{H}_A \otimes \mathcal{H}_B \cong \dots$$

☆ End

4th lecture, 13/10/23

Recap: Bra-ket notation

- vectors: $|v\rangle$ "ket v"

$$|v\rangle = \begin{pmatrix} v_0 \\ \vdots \\ v_{d-1} \end{pmatrix} = \sum_i v_i |i\rangle$$

- lin. fcn. $\langle v|$ "bra v":

$$\langle v| : |w\rangle \mapsto \langle v|w\rangle$$

$$\langle v| = \sum_i \bar{v}_i \langle i| \quad \text{if} \quad |v\rangle = \sum_i v_i |i\rangle$$

$$\langle v| = (\bar{v}_0 \; \dots \; \bar{v}_{d-1})$$

- lin. operators
  - $A = \sum_{ij} A_{ij} |i\rangle\langle j| = \begin{pmatrix} A_{00} & \dots & A_{0d-1} \\ \vdots & & \\ A_{d-10} & \dots & A_{d-1d-1} \end{pmatrix}$
  - adjoint: A^+

$$A^+ = \sum_{ij} \bar{A}_{ji} |i\rangle\langle j| = \begin{pmatrix} \bar{A}_{00} & \dots & \bar{A}_{d-10} \\ \vdots & & \\ \bar{A}_{0d-1} & \dots & \bar{A}_{d-1d-1} \end{pmatrix}$$

  - self-adjoint: $A = A^+$
  - unitary: $U^+ = U^{-1}$

- Self-adj, unitary ops are diagonalizable.
- Positivity : $A \in \mathcal{B}(\mathcal{H})$ $A \geq 0$ iff

  (1) $\langle v | A | v \rangle \geq 0 \quad \forall v \in \mathcal{H}$

  (2) $A = A^\dagger$ and $\forall$ eig. value ≥ 0

  (3) $A = XX^\dagger$
- Tensor prod : $\mathcal{H}_A \rightarrow$ basis $\{|i\rangle_A\}_{i=0}^{d-1}$

  $\mathcal{H}_B \rightarrow$ basis $\{|j\rangle_B\}_{j=0}^{d-1}$

  $\Rightarrow$ $\mathcal{H}_A \otimes \mathcal{H}_B$ is v-space w/ basis $\{|i\rangle \otimes |j\rangle \equiv |ij\rangle\}$

  Given $|v\rangle \in \mathcal{H}_A$, $|w\rangle \in \mathcal{H}_B$, their tensor prod. is

  $$|v\rangle \otimes |w\rangle = \sum_{ij} v_i w_j \, |ij\rangle$$

  <u>Ex</u>: $\mathcal{H}_A \otimes \mathcal{H}_B^* \cong Lin(\mathcal{H}_B, \mathcal{H}_A)$
- Scalar prod: $\langle \hat{v} \otimes \hat{w} | v \otimes w \rangle = \langle \hat{v} | v \rangle \cdot \langle \hat{w} | w \rangle$.

- Tensor prod. of multiple spaces:

  $\mathcal{H}_A$ has basis $\{|i\rangle\}$

  $\mathcal{H}_B$ has basis $\{|j\rangle\}$

  $\mathcal{H}_C$ has basis $\{|k\rangle\}$

  $\Big\}$ $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ has basis $\{|ijk\rangle\}$

<u>Today</u>: tensor product of matrices

quantum mechanics.

Tensor prod. of lin. operators :

- $\mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_B) \cong \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$

$$\begin{array}{ccc} \mathcal{B}(\mathcal{H}_A) & \mathcal{B}(\mathcal{H}_B) & \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \\ \cong & \cong & \cong \\ \mathcal{H}_A \otimes \mathcal{H}_A^* & \mathcal{H}_B \otimes \mathcal{H}_B^* & \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_A^* \otimes \mathcal{H}_B^* \end{array}$$

- Tensor product of two matrices:

$A \otimes B$ is def. s.t. $(A \otimes B)(|v\rangle \otimes |w\rangle) = A|v\rangle \otimes B|w\rangle$.

Applying this to the basis vectors, we obtain

$$A \otimes B = \sum_{ijkl} A_{ij} |i\rangle\langle j| \otimes B_{kl} |k\rangle\langle l| = \sum_{ijkl} A_{ij} B_{kl} |ik\rangle\langle jl| =$$

$$\begin{pmatrix} A_{00}B_{00} & \cdots & A_{0d-1}B_{0d-1} \\ \vdots & & \\ A_{d-1\,0}B_{d-1\,0} & \cdots & A_{d-1\,d-1}B_{d-1\,d-1} \end{pmatrix} = \begin{pmatrix} A_{00} \cdot \underline{\underline{B}} & \cdots & A_{0d-1} \underline{\underline{B}} \\ \vdots & & \\ A_{d-1\,0} \underline{\underline{B}} & \cdots & A_{d-1\,d-1} \underline{\underline{B}} \end{pmatrix}$$

Where the basis is ordered alphabetically:

$$v = \begin{pmatrix} v_{00} \\ v_{01} \\ \vdots \\ v_{0d-1} \\ v_{10} \\ \vdots \\ v_{d-1\,d-1} \end{pmatrix}$$

Ex.: $X \otimes Z = \begin{pmatrix} X_{00} \cdot Z & X_{01} \cdot Z \\ X_{10} \cdot Z & X_{11} \cdot Z \end{pmatrix} =$

$$= \begin{pmatrix} 0 & Z \\ Z & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

* Eig. vectors / values of elementary tensorprod.:

$(M \otimes N)(|v\rangle \otimes |w\rangle) = \mu \lambda |v\rangle \otimes |w\rangle$

if $M|v\rangle = \mu |v\rangle$ and $N|w\rangle = \lambda |w\rangle$

* General op. is of the form $\sum_i M_i \otimes N_i$, in this case no easy way to understand them,

* Tensor product of two self-adj. / unitary mx is self-adj. / unitary!

$(A \otimes B)^+ = A^+ \otimes B^+$, as

$\langle \hat{v} \otimes \hat{w} | (A \otimes B)(v \otimes w) \rangle = \langle \hat{v} \otimes \hat{w} | Av \otimes Bw \rangle$

$= \langle \hat{v} | Av \rangle \langle \hat{w} | Bw \rangle = \langle A^+ \hat{v} | v \rangle \langle B^+ \hat{w} | w \rangle$

$= \langle A^+ \hat{v} \otimes B^+ \hat{w} | v \otimes w \rangle = \langle (A^+ \otimes B^+)(\hat{v} \otimes \hat{w}) | v \otimes w \rangle$

+ works for lin. combination as well.