# Quantum framework

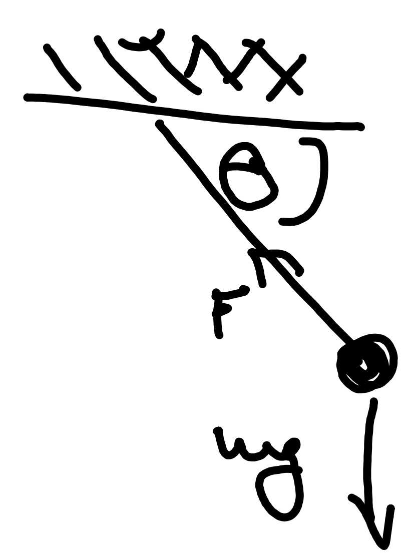
General framework how to describe "quantum" systems, let them be mechanical / electromagnetic

(+ 2 other forces a.c in play — no gravity)

Usually I say quantum mechanics.

Framework: guide what tools to use to describe systems.

For example, classical mechanical systems:




- STATE: $\theta$
- Goal: understand $\theta(t)$
- Forces. One gets $2^{nd}$ order diff. equations ($F = m \cdot a = m\ddot{x}$)
- Re-expressing as $1^{st}$ order

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \dots$$

How to do it exactly is mechanics.

The fact that we do it this way is the framework.

Other framework is <u>proba. theory</u>.
In the simplest form, the deterministic system can be described by a discreet variable, e.g. a coin can be Head / tail. When we don't have full knowledge, we use a probabilistic description: e.g. 25% Head, 75% tail.

<u>STATE</u>: probability distribution ($p$)

To describe the system, we want to know the time dependence: $p(t)$.

<u>TIME EVOLUTION</u>: Eg. some kind of master equation $\dot{p} = F(p), \ldots$

<u>MEASUREMENT</u>: reveal extra info about the system, e.g. check if coin is Head / Tail. (Can also reveal partial information) Ofc. after measuremt you have to update the proba distribution.

## Quantum framework:

- I will say q. mechanics, but:
- Discreet Degrees of Freedom (DOF), i.e. I'm not going to talk about eg. position. Instead think of a coin: Head/Tail. Physically, often these are some magnetic moments.

The constituents of framework:

→ STATES
→ TIME EVOLUTION
→ MEASUREMENTS.

There are actually 2 frameworks:

- "deterministic": or closed q. systems
- "probabilistic": or open q. systems, where we give up "full" understanding.

Even in det. q. systems measurements are

important as they change the state.
(getting out info also requires interaction)

STATES:

- Closed systems: $|v\rangle \in \mathcal{H}$, $\|v\|=1$,

  $|v\rangle \simeq e^{i\varphi}|v\rangle \quad (\varphi \in \mathbb{R})$

  Open systems: $\rho \in B(\mathcal{H})$, $\rho \geq 0$, $\mathrm{tr}(\rho)=1$.

  if $\rho = |v\rangle\langle v|$ (w/ $\|v\|=1$), then $\rho$ is called pure and it actualy describes a closed system w/ state $|v\rangle$.

  Note: $|e^{i\varphi} v\rangle\langle e^{i\varphi} v| = |v\rangle\langle v|$.

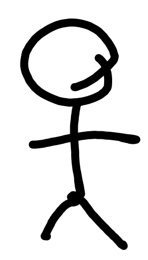
  The state $\frac{1}{d}\mathbb{1}$ is called max. mixed state, reflects 0 knowledge.

  These states reflect partial knowledge of the system. The classical probabilistic description carries over in the following sense:

if we have a system that is in a state $|\psi_i\rangle$ w/ probability $p_i$, then our description is

$$\rho = \sum_i p_i \underbrace{|\psi_i\rangle\langle\psi_i|}_{\text{state of } i^{th} \text{ possibility}}.$$

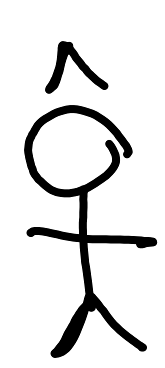
<u>Example</u>: state preparation.



50% $\to |0\rangle$

50% $\to |1\rangle$

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}\mathbb{1}$$



50% $\to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$

50% $\to \frac{1}{\sqrt{2}}\,\underbrace{(|0\rangle - |1\rangle)}_{|-\rangle} = |-\rangle$

$$\rho = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| = \frac{1}{2}\mathbb{1}.$$

- Composite systems:

(Proba theory: if 1 coin is described by $p \in \mathbb{R}^2$, $p \geq 0$, $\sum_i p_i = 1 \Rightarrow$ 2 coins are described by specifying proba of each outcome: HH, HT, TH, TT

$\Rightarrow p \in \mathbb{R}^2 \otimes \mathbb{R}^2$).

In quantum systems if system A is described on Hilbert space $\mathcal{H}_A$, system B on Hilbert space $\mathcal{H}_B$, then the composite system (2 coins) is described on $\mathcal{H}_A \otimes \mathcal{H}_B$. That is:

<u>Closed system</u>: $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$

<u>Open system</u>: $\rho \in B(\mathcal{H}_A \otimes \mathcal{H}_B) \cong B(\mathcal{H}_A) \otimes B(\mathcal{H}_B)$.

<u>TIME EVOLUTION</u>

<u>Closed systems</u>, <u>discrete timesteps</u>

(see proba theory: $p \mapsto Sp$ w/ $S$ is a stochastic matrix: doesn't change $p \geq 0$, $\sum_i p_i = 1$)

$|\psi\rangle \mapsto \mathcal{U}|\psi\rangle$ where $\mathcal{U}$ doesn't change the norm of any vector $\rightsquigarrow$ unitary.

Time ev. of indep. comp. system: $\mathcal{U}_A \otimes \mathcal{U}_B$

→ Cont. time:

$$|\psi(t)\rangle = \mathcal{U}(t)|\psi\rangle$$

$$\frac{d}{dt}|\psi(t)\rangle = \underbrace{\frac{d}{dt}\mathcal{U}(t)}_{iH(t)\mathcal{U}(t)}|\psi\rangle$$

$$= i\,H(t)\,|\psi(t)\rangle$$

## Open systems

We will only consider discrete time evolution / state transformation.

→ Linear map $T$

→ Positive to positive $T(\rho) \geq 0$ if $\rho \geq 0$

→ Keep the trace $\operatorname{Tr} T(\rho) = 1$ if $\operatorname{tr}\rho = 1$

→ If $\rho \in \mathcal{H}_A \otimes \mathcal{H}_B$, $\rho \geq 0$, then

$$(T \otimes id)(\rho) \geq 0.$$

These maps are called <u>completely positive</u> trace-preserving maps, or quantum channels.

We will learn more about them later.

☆ End.

# 5th lecture, 16/10/23

**Recap:**

- $A \in \mathcal{B}(\mathcal{H})$, $A = \sum_{ij} A_{ij} |i\rangle\langle j|$ is positive if
  - (a) $\langle v | A | v \rangle \geq 0 \quad \forall\, |v\rangle \in \mathcal{H}$
  - (b) $A = A^\dagger$ and all eig. vectors $> 0$
  - (c) $A = X X^\dagger$ for some $X \in \mathcal{B}(\mathcal{H})$

  all equivalent

- $\mathcal{H}_{A/B}$ has ONB $\{|i\rangle\}_{A/B}$, then $\mathcal{H}_A \otimes \mathcal{H}_B$ has ONB $\{|ij\rangle\}$

  $|v\rangle = \sum_i v_i |i\rangle$, $|w\rangle = \sum_i w_i |i\rangle \Rightarrow v \otimes w = \sum_{ij} v_i w_j |ij\rangle$

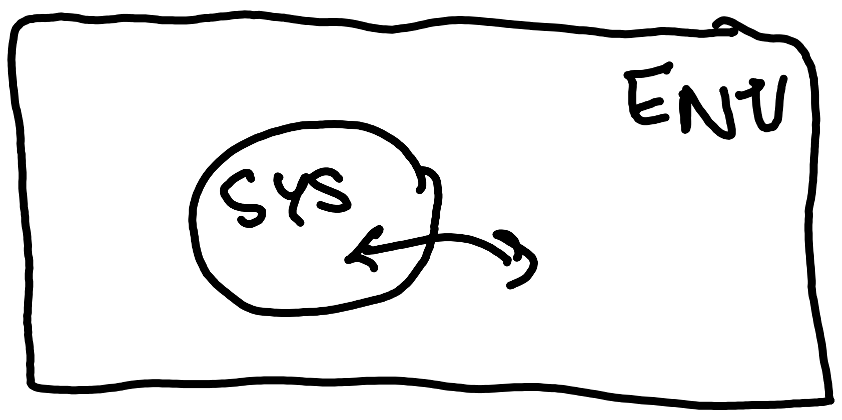
  $\mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_B)$ similar,

  $A = \sum_{ij} A_{ij} |i\rangle\langle j|$, $B = \sum_{ij} B_{ij} |i\rangle\langle j|$ :

  $$A \otimes B = \sum_{ijkl} A_{ij} B_{kl} |ik\rangle\langle jl| = \begin{pmatrix} A_{00} \underline{\underline{B}} & \cdots & A_{0d-1} \underline{\underline{B}} \\ \vdots & & \vdots \\ A_{d-1,0} \underline{\underline{B}} & \cdots & A_{d-1 d-1} \underline{\underline{B}} \end{pmatrix}$$

## Quantum theory

- Theory: mechanics/electrodynamics etc.
  - STATES : math. obj. used to describe a system
  - TIME EVOLUTION :
  - MEASUREMENT : how to get out info. from descr.
- Open/closed systems
  - → closed : usual q. mecha (isolated, pure, ..)
  - → open : interaction w/ environment, don't want to describe environment




} together closed system.

STATES :

→ Open systems : $\boxed{\text{density operators}}$ / $\boxed{\text{mixed states}}$

$$\rho \in B(\mathcal{H}), \quad \rho \geq 0, \quad \mathrm{tr}\,\rho = 1$$

→ Linearity : $p_1$ proba. $\rho_1$ and $1-p_1$ proba $\rho_2$ $\equiv$ $p_1\rho_1 + (1-p_1)\rho_2 = \rho$.

→ closed systems : special density operators : rank-1, called $\boxed{\text{pure states}}$ :

$$\rho = |\psi\rangle\langle\psi| \quad w/ \quad |\psi\rangle \in \mathcal{H}, \quad \langle\psi|\psi\rangle = 1.$$

The state of closed system is described by $|\psi\rangle \in \mathcal{H}$.

Note : as real description is $\rho$,

$|\psi\rangle$ and $e^{i\varphi}|\psi\rangle$ ($\varphi \in \mathbb{R}$) descr. same

→ comp. systems : on tensor product state.

TIME EVOLUTION

→ closed systems : $|\psi\rangle \mapsto U|\psi\rangle$, $U$ unitary.

→ open systems : $\rho \mapsto T(\rho)$, where

T is → linear

→ trace preserving

→ $(T_A \otimes id_B)(\eta_{AB}) \geq 0$ $\forall B$ and $\eta_{AB} \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$ density mx.

# MEASUREMENT:

→ You look at the q. system, get out some info, but disturb the system as well ⇒ state changes. "collapses"

Formalism is the same for open/closed syst.

**Def:** <u>POSITIVE operator valued measurement</u> (<u>POVM</u>):

Given by a set $\{M_i\}_{i=1}^{m}$, $m$: # of diff. outcomes such that

$$\sum_i M_i^{+} M_i = \mathbb{1}.$$

- Then proba of $i^{th}$ outcome:

$$p_i = \mathrm{tr}\{M_i \rho M_i^{+}\} = \mathrm{tr}\{\rho M_i^{+} M_i\}$$

(Note: $\sum_i p_i = \sum_i \mathrm{tr}\{\rho M_i^{+} M_i\} = \mathrm{tr}\{\rho\} = 1$)

and $p_i > 0$ as $M_i \rho M_i^{+} \geq 0$, and

$\mathrm{tr} = \sum$ of eig. values.

- State after measurement:

$$\rho_i = \frac{1}{p_i} M_i \rho M_i^{+}.$$

$\boxed{\text{Special case}}$ If $M_i$ are Hermitian projectors, $M_i = P_i$ w/ $P_i^2 = P_i = P_i^+$, then it's called

<u>projective measurement</u>.

In this case: $\sum_i P_i = \mathbb{1}$

- $p_i = \text{tr}\{P_i \rho P_i^+\} = \text{tr}\{\rho P_i^+ P_i\} = \text{tr}\{\rho P_i\}$
- $\rho_i = \frac{1}{p_i} \cdot P_i \rho P_i^+ = \frac{1}{p_i} P_i \rho P_i$

For pure states: $\rho = |\psi\rangle\langle\psi|$ w/ $\langle\psi|\psi\rangle = 1$

$\rightarrow \quad p_i = \text{tr}\{M_i |\psi\rangle\langle\psi| M_i^+\} = \langle\psi| M_i^+ M_i |\psi\rangle$

for proj. meas: $\langle\psi|P_i|\psi\rangle$

$\rightarrow |\psi_i\rangle = \frac{M_i|\psi\rangle}{\|M_i|\psi\rangle\|} = \frac{M_i|\psi\rangle}{\sqrt{p_i}}$

Same for proj. meas.

$\text{tr}\{|\psi\rangle\langle\psi|\} =$
$= \sum_i \langle i|\psi\rangle\langle\psi|i\rangle$
$= \sum_i \bar{\psi}_i \psi_i =$
$= \langle\psi|\psi\rangle.$
(cyclicity of tr)

We will also say "measuring an operator $O$" when $O = O^+$, and we mean: $O = \sum_i \lambda_i P_i$ spect. decomp.

- $P_i = P_i^+ = P_i^2$
- $\sum_i P_i = \mathbb{1}$

$\Big\}$ they form a measurement.

If $P_i$ are rank-1, $P_i = |\psi_i\rangle\langle\psi_i|$ $\rightsquigarrow$ meas. in a basis.

<u>We have seen</u>: quantum theory

- → How to describe STATES
- → How to describe TIME EVOLUTION
- → How to describe MEASUREMENTS.

$\boxed{\text{Prep}} \longrightarrow \text{STATE} \longrightarrow \boxed{\text{EVOLUTION}} \longrightarrow \text{STATE} \longrightarrow \boxed{\phantom{xx}}$

MEAS.

↙ proba ↘ state.

Let's see some examples before more details.

<u>$\mathcal{H} = \mathbb{C}^2$</u>: "qubits"

closed systems: $|\psi\rangle \in \mathbb{C}^2$, $\langle\psi|\psi\rangle = 1$.

mixed states: $\rho \in \mathcal{B}(\mathbb{C}^2) = M_2$

$\rho \geq 0$, $\mathrm{tr}\{\rho\} = 1$.

$$\mathrm{rank}\, \rho = \begin{cases} 1 \to \text{pure states: } \rho = |\psi\rangle\langle\psi| \\ 2 \to \text{mixed states.} \end{cases}$$

Any self-adjoint 2x2 complex matrix can be written as (see exercise)

$$\rho = \frac{\alpha}{2}\mathbb{1} + \frac{x}{2}X + \frac{y}{2}Y + \frac{z}{2}Z = \frac{1}{2}\begin{pmatrix} \alpha+z & x-iy \\ x+iy & \alpha-z \end{pmatrix}$$

$\mathrm{tr}\,\rho = 2\alpha \Rightarrow \mathrm{tr}\,\rho = 1$ iff $\alpha = \frac{1}{2}$

$$\rho = \frac{1}{2}\left(\mathbb{1} + xX + yY + zZ\right) = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}$$

$\rho \geq 0$: self-adjoint: ok.

eig. values:

$$(1+z-\lambda)(1-z-\lambda) - (x+iy)(x-iy) =$$

$$= (1-\lambda)^2 - x^2 - y^2 - z^2$$

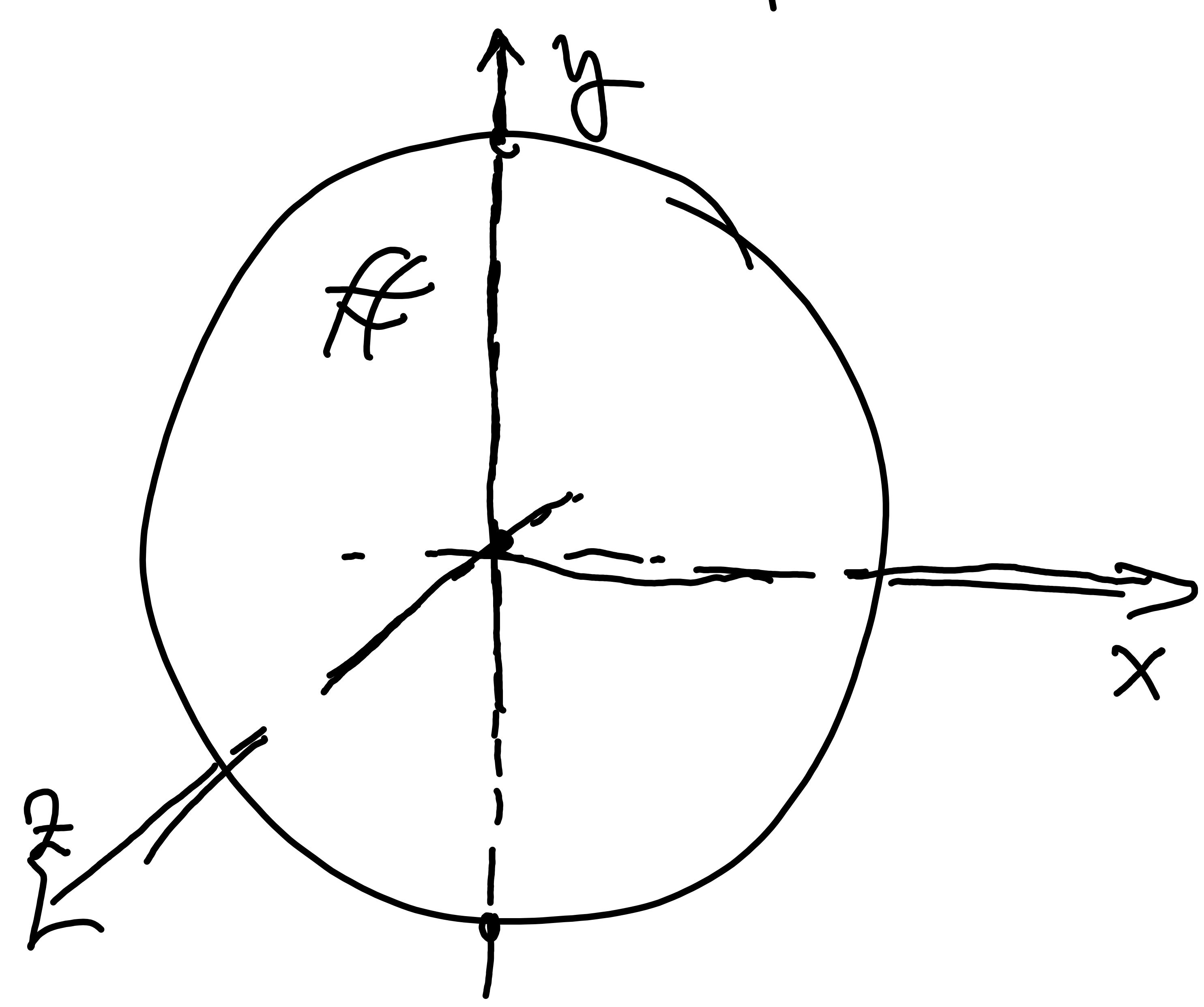
$$\lambda = 1 \pm \sqrt{x^2+y^2+z^2}$$

Therefore $\rho \geq 0$ iff $x^2+y^2+z^2 \leq 1$.

<u>Note:</u> $\rho$ is pure iff it is rank 1

iff $x^2+y^2+z^2=1$.

So one can depict 2D density matrices w/ a 3D sphere:




+ center: $\frac{1}{2}\mathbb{1}$
+ surface: pure states
+ inside: mixed states.

Special points:

N: $\frac{1}{2}(\mathbb{1}+Y)=\frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}$

S: $\frac{1}{2}(\mathbb{1}-Y)=\frac{1}{2}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}$

E: $\frac{1}{2}(\mathbb{1}+X)=\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}=|+\rangle\langle+|$

W: $\frac{1}{2}(\mathbb{1}-X)=\frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}=|-\rangle\langle-|$

F: $\frac{1}{2}(\mathbb{1}+Z)=\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}=|0\rangle\langle 0|$

B: $\frac{1}{2}(\mathbb{1}-Z)=\frac{1}{2}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}=|1\rangle\langle 1|$

## TIME EVOLUTION (pure systems):

$|\psi\rangle \mapsto U|\psi\rangle$, $U$ unitary.

Example: $H$: Hadamard $\boxed{\text{gate}}$.

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} (X + Z)$$

$$H^\dagger = H, \quad H^2 = \frac{1}{2}(X+Z)^2 = \frac{1}{2}(1 + 1 + \underbrace{XZ + ZX}_{0}) = \mathbb{1}.$$

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) = |+\rangle,$$

and $|+\rangle$ is the $+1$ eig. state of $X$.

$$H|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) = |-\rangle,$$

and $|-\rangle$ is the $-1$ eig. state of $X$.

Note: $H$ transforms btw eig. states of $X$ and $Z$, because

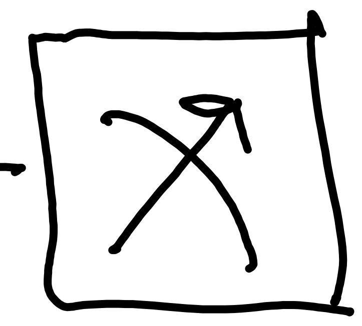
$$HXH = \frac{1}{2} (X+Z) X (X+Z) =$$

$$= \frac{1}{2} (\underbrace{XXX}_{X} + \underbrace{XXZ}_{Z} + \underbrace{ZXX}_{Z} + \underbrace{ZXZ}_{-X}) = Z.$$

or

$X = HZH^\dagger$, it diagonalizes $X$.

# Measurement :

STATE $\rightarrow$  —

$$\begin{cases} \text{outcome } 0 \rightarrow p_0 \text{ proba, } \rho_0 \text{ state} \\ \vdots \\ \text{outcome } n \rightarrow p_n \text{ proba, } \rho_n \text{ state} \end{cases}$$

"Measurement in $Z$ basis" : meas. in basis where $Z$ is diagonal : $\{|0\rangle, |1\rangle\}$.

That is, $M_0 = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$M_1 = |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

$$\left.\begin{aligned} M_0^2 &= M_0 = M_0^\dagger \\ M_1^2 &= M_1 = M_1^\dagger \end{aligned}\right\} \text{ proj. measurement}$$

$M_0^\dagger M_0 + M_1^\dagger M_1 = M_0 + M_1 = \mathbb{1}$, so they form a measurement.

There are 2 outcomes : "0" and "1"

$p_0 = \mathrm{tr}\{M_0 \rho M_0^\dagger\} = \langle 0|\rho|0\rangle$

$p_1 = \mathrm{tr}\{M_1 \rho M_1^\dagger\} = \langle 1|\rho|1\rangle$

and post-meas. state

$$\frac{M_0 \rho M_0^\dagger}{\mathrm{tr}\{M_0 \rho M_0^\dagger\}} = \frac{|0\rangle\langle 0|\rho|0\rangle\langle 0|}{\langle 0|\rho|0\rangle} = |0\rangle\langle 0| .$$

$$\frac{M_1 \rho M_1^\dagger}{\mathrm{tr}\{M_1 \rho M_1^\dagger\}} = |1\rangle\langle 1| .$$

If $\rho$ is pure: $\rho = |\psi\rangle\langle\psi|$ w/ $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, $|\alpha|^2 + |\beta|^2 = 1$, then

$$p_0 = \langle 0|\psi\rangle\langle\psi|0\rangle = |\alpha|^2 ,$$

$$p_1 = \langle 1|\psi\rangle\langle\psi|1\rangle = |\beta|^2 .$$

"Measurement in X basis": meas. in the basis where X is diagonal.

$$X = |+\rangle\langle +| + |-\rangle\langle -| ,$$

so it is the basis $\{|+\rangle, |-\rangle\}$

General operater:

$$p_+ = \langle +|\rho|+\rangle \qquad \rho_+ = |+\rangle\langle +|$$

$$p_- = \langle -|\rho|-\rangle \qquad \rho_- = |-\rangle\langle -|$$

Pure states: $\rho = |\psi\rangle\langle\psi|$, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Then

$$P_+ = |\langle + | \psi\rangle|^2 = \frac{|\alpha+\beta|^2}{2} \quad \text{, post meas. state: } |+\rangle$$

$$P_- = |\langle - | \psi\rangle|^2 = \frac{|\alpha-\beta|^2}{2} \quad \text{, post meas. state: } |-\rangle$$

Note: Sanity check:

$$P_+ + P_- = \frac{|\alpha+\beta|^2}{2} + \frac{|\alpha-\beta|^2}{2} = 1.$$

"Measurement in $Y$ basis":

→ Take $Y$

→ Write $Y = \sum_i \lambda_i P_i = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$

→ Meas. operators $P_i = |\psi_i\rangle\langle\psi_i|$

Here: $|\psi_0\rangle = (|0\rangle + i|1\rangle)\frac{1}{\sqrt{2}}$

$|\psi_1\rangle = (|0\rangle - i|1\rangle)\frac{1}{\sqrt{2}}$

Note: works for general operator $O = O^\dagger$, even if there are deg. eig. vectors!

That is, $O = O^\dagger$ can be written

$$O = \sum_i \lambda_i P_i \quad \left( = \sum_i \lambda_i \sum_j |\psi_{i,j}\rangle\langle\psi_{i,j}| \right)$$

w/ $\sum_i P_i = 1, \quad P_i^\dagger = P_i = P_i^2$.

"Measuring $O$" / "measuring in $O$-basis":

is the projective measurement $\{P_i\}$.

End ☆

# 6th lecture, 20/10/23

**Recap:** quantum mecha axioms

- "degrees of freedom": $\mathcal{H}$, composite: $\mathcal{H}_A \otimes \mathcal{H}_B$
- state: $\rho \in B(\mathcal{H})$ $\rho \geq 0$, $\mathrm{tr}\{\rho\} = 1$

  density matrix.

  [$\rho \geq 0$ iff $\rho = \rho^\dagger$ and all of its eig. values $\geq 0$.]

  pure states: rank-1, $\rho = |\psi\rangle\langle\psi|$.

  In this case we just write

  $|\psi\rangle \in \mathcal{H}$, $\| |\psi\rangle \| = 1$, $|\psi\rangle \equiv e^{i\varphi}|\psi\rangle$

  for states. <u>This is the typical description.</u>

- time evolution:
  - Closed system: unitaries

    $|\psi\rangle \mapsto U|\psi\rangle$

    $\rho \mapsto U\rho U^\dagger$
  - open system:

    $\rho \mapsto T(\rho)$ $T$ is a channel:

    - $T$ is linear
    - trace preserving. $\mathrm{tr}(T(X)) = \mathrm{tr}(X)$
    - completely positive: $\forall \mathcal{K}$ hilb. space

      $(T \otimes id_{\mathcal{K}})(\rho_{\mathcal{H}\mathcal{K}}) \geq 0$ if $\rho_{\mathcal{H}\mathcal{K}} \geq 0$.

    (We'll see later why)

– Measurement: outcomes, probabilities, post-meas. state

A meas. w/ $\underline{n}$ outcomes is given by
$\{M_i\}_{i=1}^{n}$ s.t. $\sum_i M_i^+ M_i = \mathbb{1}$.
Probability of outcome $i$ is
$$p_i = \mathrm{tr}\{M_i \rho M_i^+\} = \mathrm{tr}\{\rho M_i^+ M_i\}$$
Post-meas. state is
$$\rho_i = \frac{1}{p_i} M_i \rho M_i^+.$$

Today: more about meas. and states.

If you don't want to describe post-meas. states, then it is enough to give the operators $E_i = M_i^+ M_i$, i.e. this meas. is given by $\{E_i\}_{i=1}^{n}$ s.t. $E_i \geq 0$, $\sum_i E_i = \mathbb{1}$, and $p_i = \mathrm{tr}\{\rho E_i\}$.
$\rightsquigarrow$ positive operators $\rightsquigarrow$ <u>P</u>ositive <u>O</u>perator <u>V</u>alued <u>M</u>easurement, POVM,

- Measuring $O = O^+$: $O = \sum_\lambda \lambda P_\lambda \rightarrow \{P_\lambda\}$ POVM
$\langle O \rangle = \sum \lambda p_\lambda = \sum_\lambda \lambda \, \mathrm{tr}(\rho P_\lambda) = \mathrm{tr}(\rho O)$.

## Convex structure of the state space:

Let $V$ be a real vector space,

$S \subseteq V$ a subset. We say that $S$ is convex if $\forall\, v, w \in S$ and $\lambda \in [0,1]$

$$\lambda v + (1-\lambda) w \in S.$$

Thm: The set of density matrices

$$S = \{ \rho \in \mathcal{B}(\mathcal{H}) \mid \operatorname{tr}(\rho) = 1 \text{ and } \rho \geq 0 \}$$

is convex.

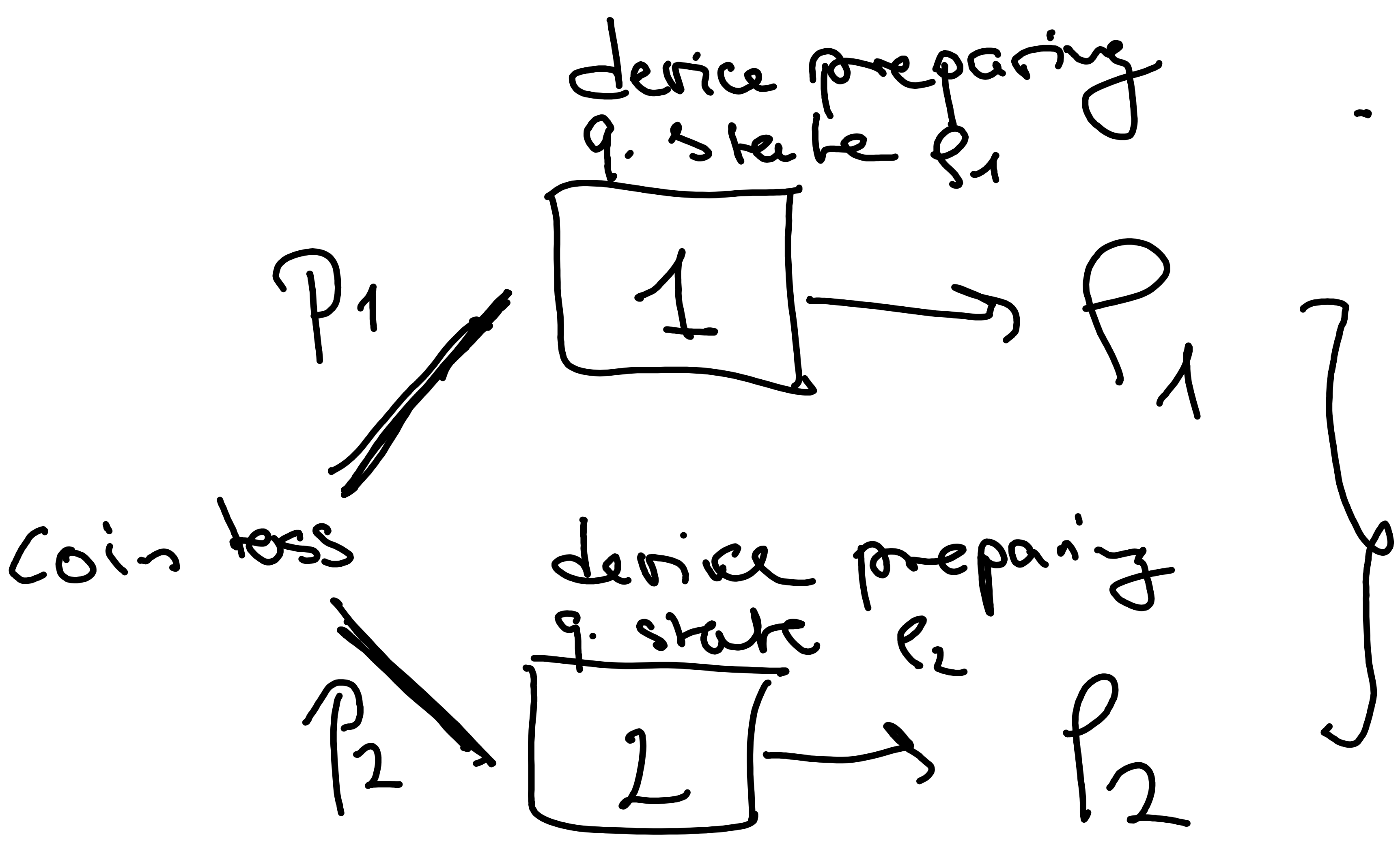
Proof: Let $p \in [0,1]$; $\rho, \sigma \in S$. Then

$\operatorname{tr}\{p\rho + (1-p)\sigma\} = p \operatorname{tr}\{\rho\} + (1-p)\operatorname{tr}\{\sigma\} = p + 1 - p = 1.$

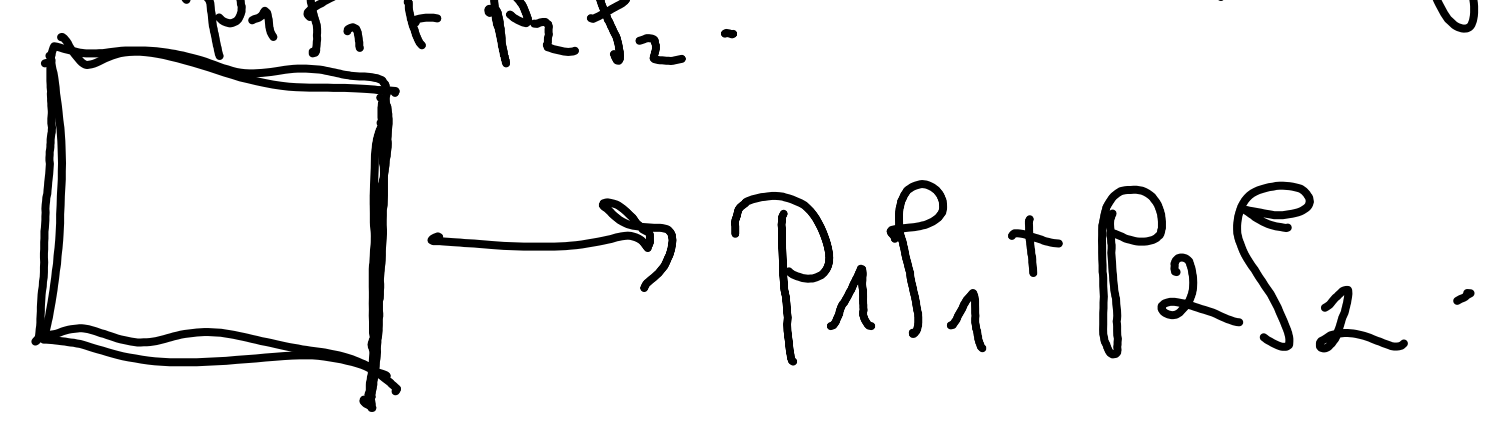
As $p \geq 0$ and $1-p \geq 0$ $\quad p\rho + (1-p)\sigma \geq 0$. $\quad \square$

This convex structure is exactly the same as what you expect from probabilistic interpretation:

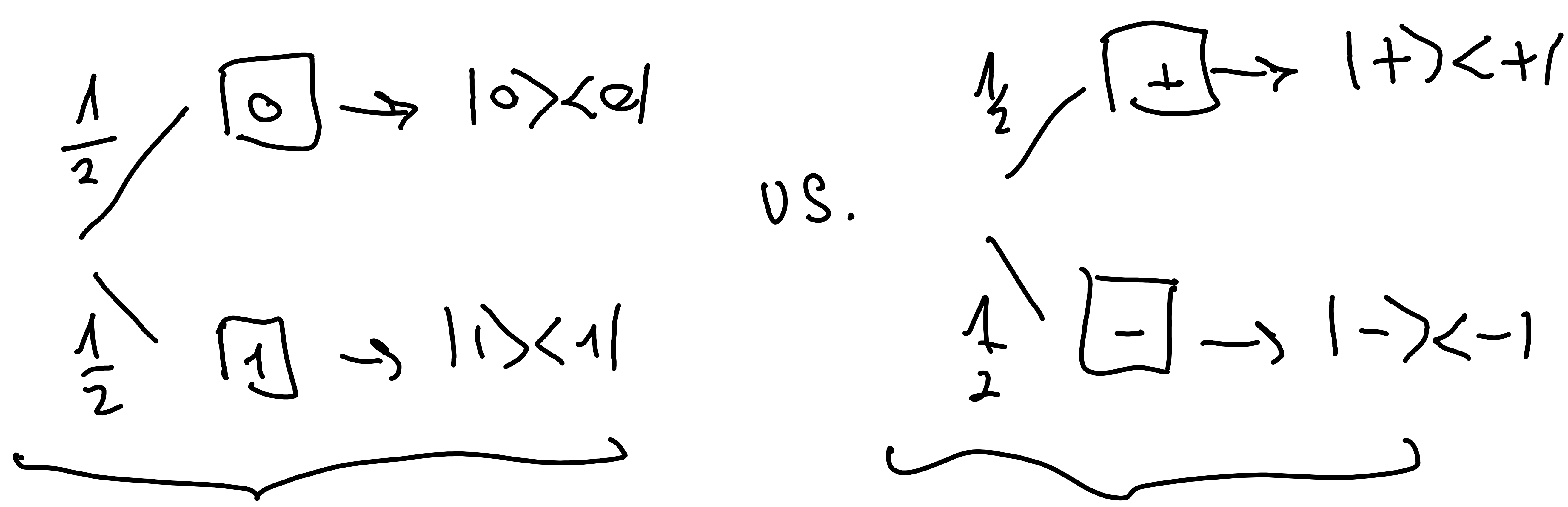
# On the probabilistic interpretation of mixed states:




This process is equivalent to a single device preparing $p_1\rho_1 + p_2\rho_2$.




## For example:




result: $\frac{1}{2}\mathbb{1}$ result: $\frac{1}{2}\mathbb{1}$

Can't differentiate!

This probabilistic interpretation helps to understand why one uses POVMs (w/o post-measurement description) to describe measurements:

What do we want from a measurement?

"outcomes" w/ probabilities.

$M: \rho \mapsto p \in \mathbb{R}^n,\ p \geq 0,\ \sum_i p_i = 0$

We want M to keep this complex str.:

$M(p_1 \rho_1 + p_2 \rho_2) = p_1 \cdot p + p_2 \cdot p$,

So we want M to be <u>linear</u>.

Any lin. op. is of the form

$p_i = \mathrm{tr}(\rho E_i)$, $E_i \in \mathcal{B}(\mathcal{H})$.

We want $p_i \in \mathbb{R}$, $p_i \geq 0$, so

$\mathrm{tr}(\rho E_i) \geq 0 \ \forall \rho$, in part. pure states: $\rho = |\psi\rangle\langle\psi|$:

$\mathrm{tr}(|\psi\rangle\langle\psi| E_i) = \langle\psi|E_i|\psi\rangle \geq 0$, so $E_i \geq 0$.

Finally, $\sum_i p_i = 1 = \sum_i \mathrm{tr}(\rho E_i)$ implies

$\sum_i E_i = 1$ (e.g. through setting $\rho$ pure, $\langle\psi|\sum_i E_i|\psi\rangle = \langle\psi|\psi\rangle\ \forall|\psi\rangle$)

The density operator $\rho$ reflects all knowledge about the system (the same way as a proba. distr. reflects all knowlede about a cl. system); and this descr. is the most compressed.

Thm: $\rho$ is uniquely determined by

(the exp. value of) all possible meas. decr. by $O = O^+$.

Proof: $V = \{ A \in B(\mathcal{H}) \mid A = A^+ \}$ is a real v-space:

if $\lambda, \mu \in \mathbb{R}$, $A, B \in V$ then $\lambda A + \mu B \in V$.

It has scalar product

$$(A, B) = \mathrm{tr}(A^+ B) = \mathrm{tr}(AB).$$

Take an ONB : $(O_i, O_j) = \delta_{ij}$

- $\rho = \sum_i \lambda_i O_i$
- $\langle O_i \rangle = \mathrm{tr}\{\rho O_i\} = \sum_j \lambda_j \mathrm{tr}(O_i O_j) = \lambda_i$

So

$$\rho = \sum_i \langle O_i \rangle \cdot O_i .$$

That is $\rho$ uniquely encodes all info about the system. Compare: $|\psi\rangle \in \mathcal{H}$, $\|\psi\| = 1$.

$$|\psi\rangle \equiv \nu \cdot |\psi\rangle \quad \text{for } |\nu| = 1.$$

**Def**: a set $\{(p_i, \rho_i)\}$ with $p_i \geq 0$, $\sum_i p_i = 1$ and $\rho_i$ density matrices are called an <u>ensemble</u>.

If $\rho = \sum_i p_i \rho_i$, then $\{(p_i, \rho_i)\}$ is an ensemble decomposition of $\rho$.

**Note**: Meas. outcomes w/ post-meas. states form an ensemble:

$p_i = \mathrm{tr}\{\rho M_i^\dagger M_i\} \Rightarrow$ proba distr.

$\rho_i = \frac{1}{p_i} M_i \rho M_i^\dagger \Rightarrow \rho_i$ are density matrices.

In general, it is not an ensemble decomp. of $\rho$;

E.g. $M_0 = |0\rangle\langle 0|$, $M_1 = |1\rangle\langle 1|$ in $\mathbb{C}^2$

$\rho = \frac{1}{2}(\mathbb{1} + aX + bY + cZ) \Rightarrow$ $p_0 = \frac{1}{2}(1+c)$ $\rho_0 = |0\rangle\langle 0|$

$p_1 = \frac{1}{2}(1-c)$ $\rho_1 = |1\rangle\langle 1|$

$p_0\rho_0 + p_1\rho_1 = \frac{1}{2}(1 + cZ)$.

There are different decompositions and they are not inherent to $\rho$: can't pick one / distinguish them.

Given any mixed state, one can imagine it as a mixture of pure states, e.g.

$$\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i| \quad \text{eigen decomposition.}$$

$\rightarrow$ this might not be unique:

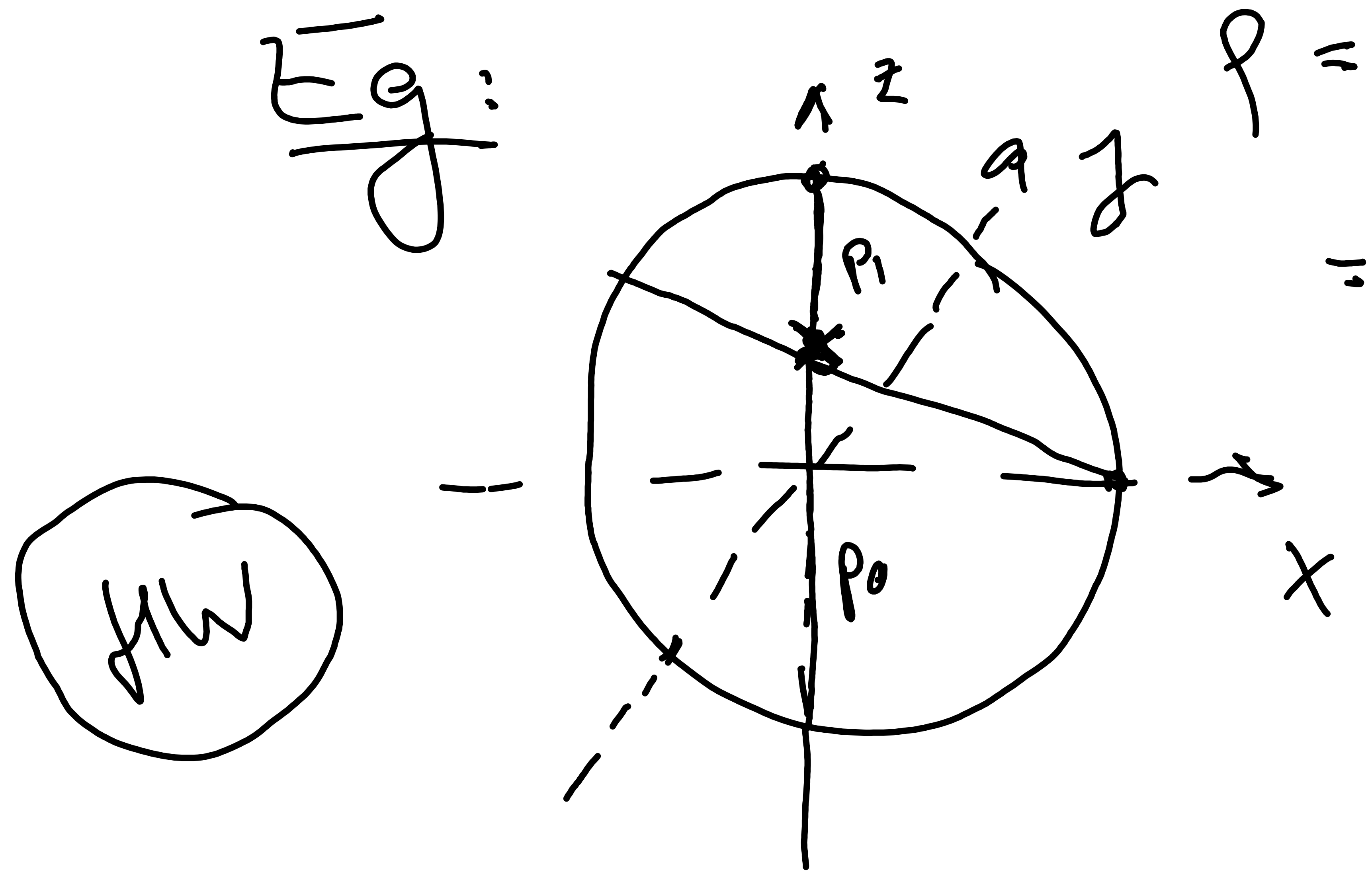
$$\frac{1}{2}\mathbb{1} = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) = \frac{1}{2}\left(|+\rangle\langle +| + |-\rangle\langle -|\right)$$

Interestingly, one can imagine $\rho$ as a mixture of other pure states as well: there are examples for

$$\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i| \,, \quad \text{where } \lambda_i, |\psi_i\rangle \text{ are}$$

<u>not</u> eig. vector/value, esp. <u>not orthogonal</u>

<u>Eg:</u>




$$\rho = p_0|0\rangle\langle 0| + p_1|1\rangle\langle 1|$$
$$= \mu|+\rangle\langle +| + \left(\rho - \mu|+\rangle\langle +|\right)$$

take max. of $\mu$ s.t.

$$\rho - \mu|+\rangle\langle +| \geq 0 .$$

It is then rank-1, thus we have another decomp.

Thm: $\rho = \sum_{i=1}^{n} p_i |\psi_i\rangle\langle\psi_i| = \sum_{i=1}^{m} q_i |\phi_i\rangle\langle\phi_i|$

w/ $p_i > 0$, $\sum_i p_i = 1$

$q_i > 0$, $\sum_i q_i = 1$

If and only if there is a matrix $U$ s.t.

(1) $\sqrt{p_i}|\psi_i\rangle = \sum_j U_{ij} \sqrt{q_j} |\phi_j\rangle$

and

(2) $\sqrt{q_i} |\phi_i\rangle = \sum_j (U^\dagger)_{ij} \sqrt{p_j} |\psi_j\rangle$

☆ End

# $7^{th}$ lecture, 23/20/23

**Recap**: QM

- DEGREES OF FREEDOM: $\mathcal{H}$, $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ composite system
- STATES: $\rho \in \mathcal{B}(\mathcal{H})$, $\rho \geq 0$, $\mathrm{tr}(\rho) = 1$
- TIME EVOLUTION: $U \in \mathcal{B}(\mathcal{H})$ unitary / T TPCP
- MEASUREMENTS: $\{\Pi_i\}_{i=0}^{n}$, $\sum_i \Pi_i^\dagger \Pi_i = \mathbb{1}$

  $p_i = \mathrm{tr}\{\rho\, \Pi_i^\dagger \Pi_i\}$

  $\rho_i = \frac{1}{p_i} \Pi_i \rho \Pi_i$

  w/o post-meas state: $\{\Pi_i^\dagger \Pi_i\}$

We have seen that the state space

$$S(\mathcal{H}) = \{\rho \in \mathcal{B}(\mathcal{H}) \mid \rho \geq 0,\ \mathrm{tr}\rho = 1\}$$

is a convex set. This set has a probabilistic interpretation:

$$\rho = \sum_i p_i \rho_i$$

w/ $\rho_i \in S(\mathcal{H})$, $p_i \geq 0$, $\sum_i p_i = 0$ is interpreted as a state that is w/ proba $p_i$ is in the state $\rho_i$.

This is also called an ensemble decomposition of $\rho$.

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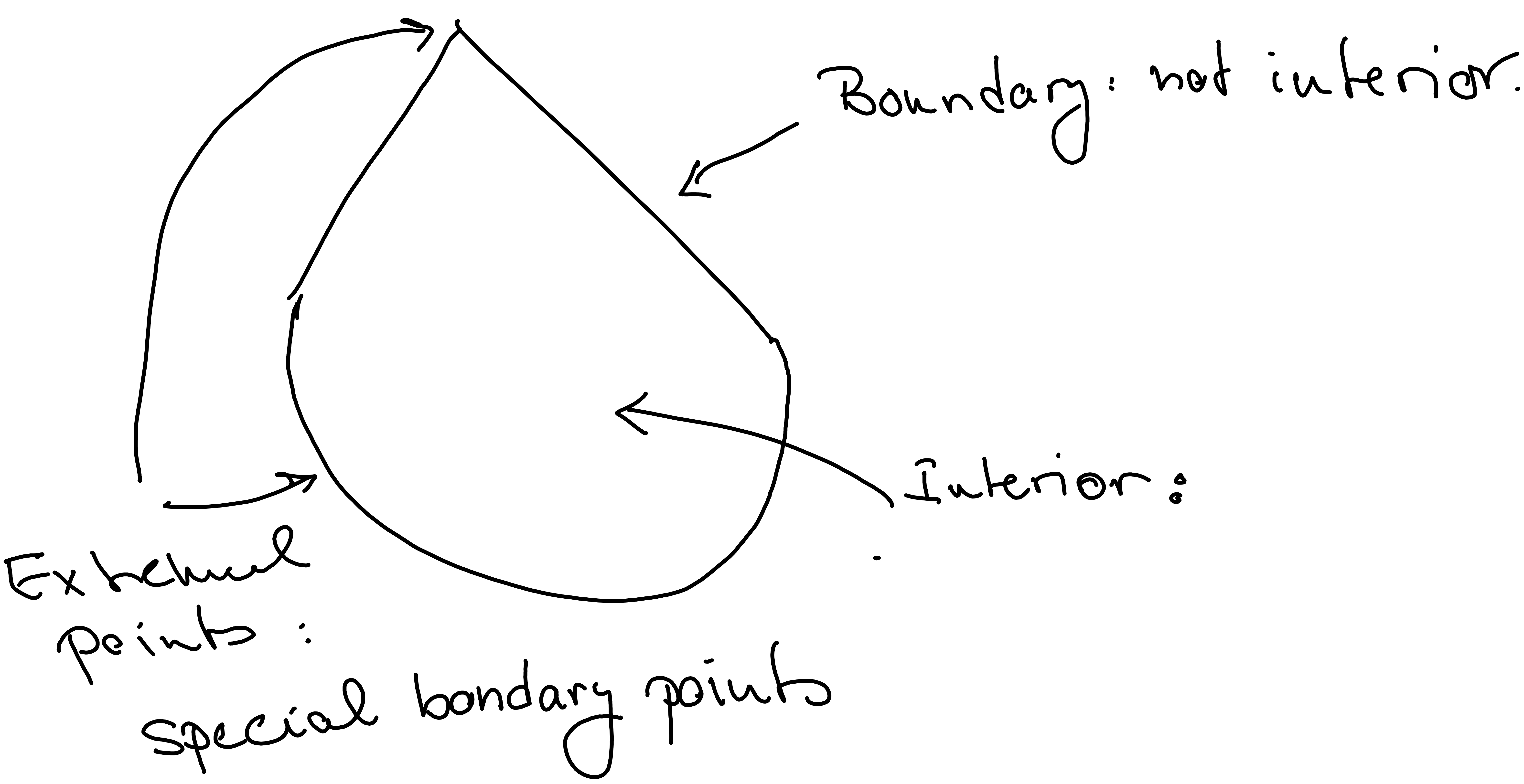
How does this convex set look like?
The difference of two density matrices is in

$$V_0 := \{ X \in \mathcal{B}(\mathcal{H}) \mid X^\dagger = X,\ \operatorname{tr} X = 0 \}.$$

This is a $D^2-1$ dimensional <u>real</u> v-space.
(Eg. for $\mathbb{C}^2$ $D^2-1=3$, Bloch sphere)




* Interior: $\forall X \in V_0\ \exists\, \varepsilon > 0$ s.t.

$$\rho + \varepsilon X \in S(\mathcal{H}).$$

Interior points can always be written as convex combination: $\forall x > 0 \ \exists \varepsilon > 0$ s.t.

$\rho + \varepsilon X \geq 0$ and $\rho - \varepsilon x > 0 \Rightarrow \rho = \frac{1}{2}(\rho + \varepsilon X + \rho - \varepsilon X) =$

$$= \frac{tr(\rho+\varepsilon x)}{2} \frac{\rho+\varepsilon x}{tr(\rho+\varepsilon x)} + \frac{tr(\rho-\varepsilon x)}{2} \frac{\rho-\varepsilon x}{tr(\rho-\varepsilon x)} .$$

Interior points are full-rank matrices:

if $\sup_{|v\rangle = 1} |\langle v|X|v\rangle| = \lambda$ and $\inf_{\||v\rangle\| = 1} \langle v|\rho|v\rangle = \mu$

($\mu > 0$ iff $\rho$ is full-rank!), then

$$\langle v|\rho - \frac{\mu}{\lambda} X|v\rangle = \langle v|\rho|v\rangle - \frac{\mu}{\lambda}\langle v|X|v\rangle$$

$\geq \mu - \frac{\mu}{\lambda}\lambda = 0$, so, if $X \in V_0$, then

$\rho - \frac{\mu}{\lambda} X \in S(\mathcal{H})$.

* <u>Boundary</u>: $\rho$ <u>not</u> full-rank, $\Rightarrow$ there is a dir. that brings us out of $S(\mathcal{H})$: if $\langle v|\rho|v\rangle = 0$, then $\rho - \lambda|v\rangle\langle v| \notin S(\mathcal{H})$ for any $\lambda > 0$. $\rho$ might still be decomposed into lin. comb. of for

example: $\rho \in \mathcal{B}(\mathbb{C}^3)$

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix}.$$

In 2D this is not the case, $\forall$ boundary is :

× <u>Extremal points</u> : can't be written as a non-trivial convex combination:

if $\rho = p\rho_1 + (1-p)\rho_2$ with $p \in [0,1]$ and $\rho_1, \rho_2 \in S(\mathcal{H})$, then

$$(p = 0 \text{ or } \rho_1 = \rho) \text{ and } (p = 1 \text{ or } \rho_2 = \rho).$$

<u>Lemma</u>: extremal points are exactly the pure states.

<u>Proof</u>: $\boxed{\Rightarrow}$ if $\rho$ is not pure, then it has an eig. vector $|\psi\rangle$ w/ eig value $0 < \lambda < 1$, so

$$\rho = \underset{1 > 1-\lambda > 0}{\underset{\uparrow}{(1-\lambda)}} \cdot \underbrace{\frac{1}{1-\lambda}\left(\rho - \lambda|\psi\rangle\langle\psi|\right)}_{\text{density mx}} + \underset{\lambda > 0}{\underset{\uparrow}{\lambda}} \cdot \underset{\text{density mx} \neq \rho}{\underset{\uparrow}{|\psi\rangle\langle\psi|}}$$

$\boxed{\Leftarrow}$ If $\rho = |\psi\rangle\langle\psi|$, then if

$$\rho = p\sigma_1 + (1-p)\sigma_2 \text{, then}$$

$$1 = \langle\psi|\rho|\psi\rangle = p\langle\psi|\sigma_1|\psi\rangle + (1-p)\langle\psi|\sigma_2|\psi\rangle,$$

and $0 \leq \langle\psi|\sigma_1|\psi\rangle \leq \lambda_{max} \leq 1$ ( ! Only for positive ) HW

So

$$1 = p\langle\psi|\sigma_1|\psi\rangle + (1-p)\langle\psi|\sigma_2|\psi\rangle$$

$$\geq p\,1 + (1-p)\,1 = 1$$

$\Rightarrow$ equality holds

$\Rightarrow$ $p = 0$ or $\langle\psi|\sigma_1|\psi\rangle = 1$, $\sigma_1 = \rho$

and $p = 1$ or $\langle\psi|\sigma_2|\psi\rangle = 1$, $\sigma_2 = \rho$. $\square$

So extremal points are pure states.

<u>Fact</u>: in a convex set (finite dim) every point can be decomposed into convex comb. of extremal points.

For us: $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. There might be multiple such decomp.

Another way to think about this:

Create a big matrix with

$$\Psi = \left( \sqrt{p_1}\,|\psi_1\rangle \,\middle|\, \sqrt{p_2}\,|\psi_2\rangle \,\middle|\, \dots \,\middle|\, \sqrt{p_n}\,|\psi_n\rangle \right)$$

column vector

$$= \sum_{i=1}^{n} \sqrt{p_i}\, |\psi_i\rangle\langle i| \in \mathrm{Lin}(\mathbb{C}^n, \mathcal{H})$$

Then

$$\Psi \Psi^\dagger = \sum_{i=1}^{n} \sqrt{p_i p_j}\, |\psi_i\rangle\langle i|j\rangle \langle\psi_j|$$

$$= \sum_i p_i\, |\psi_i\rangle\langle\psi_i| = \rho$$

So diff. ensemble decomposition = different ways to decompose

$$\rho = X X^\dagger.$$

Thm: Let $\{(p_i, |\psi_i\rangle\langle\psi_i|)\}_{i=1}^{n}$ and $\{(q_i, |\phi_i\rangle\langle\phi_i|)\}_{i=1}^{m}$ be two ensembles. Then

$$\sum_{i=1}^{n} p_i |\psi_i\rangle\langle\psi_i| = \sum_{i=1}^{m} q_i |\phi_i\rangle\langle\phi_i|$$

iff $\exists\, \mathcal{U} \in \mathrm{Lin}(\mathbb{C}^n, \mathbb{C}^m)$ s.t.

(1) $\mathcal{U}\mathcal{U}^+$ and $\mathcal{U}^+\mathcal{U}$ are projectors

(2) $\sqrt{p_i}\,|\psi_i\rangle = \sum_j \mathcal{U}_{ij} \sqrt{q_j}\,|\phi_j\rangle$

Proof: $\boxed{\Longleftarrow}$ Define

$$\Psi = \sum_{i=1}^{n} \sqrt{p_i}\,|\psi_i\rangle\langle i| \in \mathcal{L}in(\mathbb{C}^n, \mathcal{H})$$

$$\Phi = \sum_{i=1}^{m} \sqrt{q_i}\,|\phi_i\rangle\langle i| \in \mathcal{L}in(\mathbb{C}^m, \mathcal{H})$$

Then the condition (2) means

$$\Psi = \sum_i \sqrt{p_i}\,|\psi_i\rangle\langle i| = \sum_{ij} \mathcal{U}_{ij} \sqrt{q_j}\,|\phi_j\rangle\langle i|$$

$$= \underbrace{\sum_{ij} \sqrt{q_j}\,|\phi_j\rangle\langle j|}_{\Phi} \cdot \underbrace{\sum \mathcal{U}_{ik}\,|k\rangle\langle i|}_{\mathcal{U}}$$

$$= \Phi\,\mathcal{U}.$$

We can then notice that

$$\Psi\Psi^\dagger = \Phi \mathcal{U}\mathcal{U}^\dagger \Phi^\dagger \leq \Phi\Phi^\dagger,$$

as $\mathcal{U}\mathcal{U}^\dagger$ is an orthogonal projecter, and thus $\mathcal{U}\mathcal{U}^\dagger \leq \mathbb{1}$.

(Remember: we write $A \leq B$ iff $A$ and $B$ are self-adjoint and $B-A$ is a positive operater.
Remember: $A \leq B \Rightarrow XAX^\dagger \leq XBX^\dagger$ for any $X$)

Let us note now that the trace of both $\Psi\Psi^\dagger$ and $\Phi\Phi^\dagger$ is one. For example,

$$\mathrm{tr}(\Psi\Psi^\dagger) = \mathrm{tr}\Big(\sum_{ij} \sqrt{p_i}\, |\psi_i\rangle\langle i|j\rangle\langle\psi_j|\sqrt{p_j}\Big) =$$

$$= \mathrm{tr}\sum_i p_i |\psi_i\rangle\langle\psi_i|) = \sum_i p_i = 1.$$

But $\Psi\Psi^\dagger \leq \Phi\Phi^\dagger$, thus, by definition, $\Phi\Phi^\dagger - \Psi\Psi^\dagger$ is a positive operater.

As $\mathrm{tr}(\Phi\Phi^\dagger) = \mathrm{tr}(\Psi\Psi^\dagger) = 1$, it is also tr-less.

This implies:

$$\Phi\Phi^\dagger - \Psi\Psi^\dagger = 0,$$

i.e. $\Psi\Psi^\dagger = \Phi\Phi^\dagger$, or

$$\sum_i p_i |\psi_i\rangle\langle\psi_i| = \sum_i q_i |\phi_i\rangle\langle\phi_i|.$$

$\boxed{\Rightarrow}$ Let $\rho = \sum_{k=1}^{r} \lambda_k |x_k\rangle\langle x_k|$ be an eigen decomp of $\rho$, with $|x_k\rangle$'s ON and $\lambda_k > 0$.

Let $|x\rangle$ be an eig. vector w/ 0 eig. value, then

$$0 = \langle x|\rho|x\rangle = \sum_i p_i \langle x|\psi_i\rangle\langle\psi_i|x\rangle$$

$$= \sum_i \underbrace{p_i}_{\overset{\vee}{0}} \| \langle x|\psi_i\rangle \|^2$$

Therefore $\forall i$ $\langle x|\psi_i\rangle = 0$.

This implies that $|\psi_i\rangle$ is a lin. comb. of the eig vectors w/ non-zero eig. value of $\rho$,

$$\sqrt{p_i}|\psi_i\rangle = \sum_k V_{ik} \sqrt{\lambda_k}|x_k\rangle.$$

Notice now that

$$\sqrt{\lambda_k}|x_k\rangle = \frac{1}{\sqrt{\lambda_k}} \rho|x_k\rangle = \sum_i \frac{p_i}{\sqrt{\lambda_k}} |\psi_i\rangle\langle\psi_i|x_k\rangle$$

$$= \sum_i \sqrt{p_i}|\psi_i\rangle \cdot \sum_j \overline{V_{ij}} \frac{\sqrt{\lambda_j}}{\sqrt{\lambda_k}} \langle x_j|x_k\rangle$$

$$= \sum_i (V^{\dagger})_{ji} \sqrt{p_i}|\psi_i\rangle.$$

<u>Finally:</u> $\phi$ can be expressed similarly w/ $\chi_i$:

$$\sqrt{q_i}|\phi_i\rangle = \sum_j W_{ij} \sqrt{\lambda_j} |\chi_j\rangle$$

$$\sqrt{\lambda_i}|\chi_i\rangle = \sum_j (W_{ij})^\dagger \sqrt{q_j} |\phi_j\rangle$$

Therefore

$$\sqrt{q_i}|\phi_i\rangle = \sum_j W_{ij} \sqrt{\lambda_j} |\chi_j\rangle = \sum_{jk} W_{ij} (V^\dagger)_{jk} \sqrt{p_k} |\psi_k\rangle$$

$$\sqrt{p_i}|\psi_i\rangle = \sum_j V_{ij} \sqrt{\lambda_j} |\chi_j\rangle = \sum_{jk} V_{ij} (W^\dagger)_{jk} \sqrt{q_k} |\phi_k\rangle$$

So $\quad U = VW^\dagger, \; U^\dagger = WV^\dagger.$ $\square$

We have thus seen that one can write $\rho$ as an ensemble of pure states and can relate all such ensemble decompositions to eachother.

☆ End

8th lecture: 27/10/23

* trace is cyclic
* positivity: $\geq$

<u>QM systems</u>:

- DOF: $\mathcal{H}$ Hilbert space,
- States: $\rho \in \mathcal{B}(\mathcal{H})$, $\rho \geq 0$, $\mathrm{tr}\{\rho\} = 1$.
- Convex structure: if $\rho_1, \dots, \rho_n$ are density matrices, $p_1, \dots, p_n$ is a proba. distribution ($p_i \geq 0$, $\sum_i p_i = 1$), then $\sum_i p_i \rho_i$ is also density matrix.

  The set $\{(p_i, \rho_i)\}_{i=1}^{n}$ is called an ensemble. Probabilistic interpretation.

- $$\sum_{i=1}^{n} p_i |\psi_i\rangle\langle\psi_i| = \sum_{i=1}^{m} q_i |\phi_i\rangle\langle\phi_i|$$

  w/ $p_i, q_i \geq 0$, $\sum_i p_i = \sum_i q_i = 1$, $\|\phi_i\| = 1$, $\|\psi_i\| = 1$

  iff $\exists\, \mathcal{U} \in \mathbb{C}^n \times \mathbb{C}^m$ s.t.

  - $\sqrt{p_i}\,|\psi_i\rangle = \sum_j \mathcal{U}_{ij} \sqrt{q_j}\,|\phi_i\rangle$
  - $\mathcal{U}\mathcal{U}^+$ and $\mathcal{U}^+\mathcal{U}$ are projectors.

Break @ 12:15.

End @ 13:00.

# Composite systems

Given system $A$ w/ DoF $\mathcal{H}_A$, $B$ w/ DoF $\mathcal{H}_B$, the composite system has DoF $\mathcal{H}_A \otimes \mathcal{H}_B$. Why? Same as proba. theory! 1 coin H/T, then 2 coins can have HH/HT/TH/TT, so $p_{AB} \in \mathbb{R}^2 \otimes \mathbb{R}^2$.

So states in the composite system are:

$$\rho_{AB} \in B(\mathcal{H}_A \otimes \mathcal{H}_B), \quad \rho_{AB} \geq 0, \quad \mathrm{tr}\{\rho_{AB}\} = 1.$$

We have seen that density matrix $\simeq$ probability distribution

In classical PT, one can discard part of a composite system: one can take <u>marginal</u>.

Two coins' joint proba. distr: $p_{00}, p_{01}, p_{10}, p_{11}$

Now toss the two coins, but don't look at the second. What's the probability distr. of the first coin?

$p_0 = p_{00} + p_{01}$ and $p_1 = p_{10} + p_{11}$

In QM very similar thing happens:

Given $\rho_{AB} \in \mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_B)$ we can construct its marginal, called reduced density matrix/operator on A:

$$\rho_A = \mathrm{tr}_B(\rho_{AB}) := (\mathbb{1} \otimes \mathrm{tr})(\rho_{AB}),$$

where $\rho_{AB} \in \mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_B)$ is viewed as a vector, and $\mathbb{1} \in \mathcal{B}(\mathcal{B}(\mathcal{H}_A))$ and $\mathrm{tr}_B \in \mathcal{L}in(\mathcal{B}(\mathcal{H}_B), \mathbb{C})$ as lin. op.

Concretely, if

$$\rho = \sum_{ijkl} \rho_{ijkl} |ij\rangle\langle kl|, \text{ then}$$

$$\rho_A = \sum_{ijkl} \rho_{ijkl} |i\rangle\langle k| \cdot \mathrm{tr}(|j\rangle\langle l|)$$

$$= \sum_{ikj} \rho_{ijkj} |i\rangle\langle k|$$

Or equivalently, if $\rho = \sum_i X_i \otimes Y_i$, then $\rho_A = \sum_i X_i \cdot \mathrm{tr}(Y_i)$.

Remark: result does not depend on the particular decomp.

Example:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$\rho = \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|)$$

$$= \frac{1}{2}\Big(|00\rangle\langle 00| + |11\rangle\langle 00| + |00\rangle\langle 11|$$

$$+ |11\rangle\langle 11|\Big) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\rho_A = \frac{1}{2}\Big( |0\rangle\langle 0| \operatorname{tr}(|0\rangle\langle 0|) +$$

$$+ |1\rangle\langle 0| \operatorname{tr}(|1\rangle\langle 0|) +$$

$$+ |0\rangle\langle 1| \operatorname{tr}(|0\rangle\langle 1|)$$

$$+ |1\rangle\langle 1| \operatorname{tr}(|1\rangle\langle 1|) \Big)$$

$$= \frac{1}{2}\big(|0\rangle\langle 0| + |1\rangle\langle 1|\big) = \frac{1}{2}\mathbb{1} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Same way,

$$\rho_B = (\operatorname{tr} \otimes id)(\rho) = \frac{1}{2}\big( \operatorname{tr}(|0\rangle\langle 0|)\, |0\rangle\langle 0| + \operatorname{tr}(|1\rangle\langle 0|)\, |1\rangle\langle 0| +$$

$$+ \operatorname{tr}(|0\rangle\langle 1|)\, |0\rangle\langle 1| + \operatorname{tr}(|1\rangle\langle 1|)\, |1\rangle\langle 1|\big) = \frac{1}{2}\mathbb{1} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note: $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$ has same marginals / reduced densities, $\rho_B = \rho_A$. In general, this is not true. But:

Thm: Given a pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, the reduced densities

$\rho_A = \mathrm{tr}_B |\psi\rangle\langle\psi|$ and $\rho_A = \mathrm{tr}_A |\psi\rangle\langle\psi|$

have the same spectrum* (ex. for 0):

$$\mathrm{Spec}\{\rho_A\} \cup \{0\} = \mathrm{Spec}\{\rho_B\} \cup \{0\}.$$

*: the collection of all eigenvalues.

Proof: Consider the Schmidt decomposition of $|\psi\rangle$:

$$|\psi\rangle = \sum_{i=1}^{r} \lambda_i \, |\phi_i\rangle_A \otimes |\chi_i\rangle_B ,$$

where $\{|\phi_i\rangle\}$ are ON and $\{|\chi_i\rangle\}$ are ON, $\lambda_i > 0$.

Then

$$|\psi\rangle\langle\psi| = \sum_{ij=1}^{r} \lambda_i \lambda_j \, |\phi_i\rangle\langle\phi_j| \otimes |\chi_i\rangle\langle\chi_j|$$

and

$$\rho_A = \mathrm{tr}_B |\psi\rangle\langle\psi| = \sum_{ij=1}^{r} \lambda_i \lambda_j \, |\phi_i\rangle\langle\phi_j| \underbrace{\langle\chi_j|\chi_i\rangle}_{\delta_{ij}}$$

$$= \sum_{i=1}^{r} \lambda_i^2 \, |\phi_i\rangle\langle\phi_i|$$

Similarly,

$$\rho_B = \sum_{i=1}^{r} \lambda_i^2 \, |\chi_i\rangle\langle\chi_i| .$$

As $\{|\phi_i\rangle\}$ and $\{|\chi_i\rangle\}$ are orthonormal, this is the spectral decomposition:

$$\rho_A|\phi_i\rangle = \sum_j \lambda_j^2 |\phi_j\rangle\langle\phi_j|\phi_i\rangle = \lambda_i^2 |\phi_i\rangle.$$

Therefore the spectrum of $\rho_A$ is $\{\lambda_i^2\}_{i=1}^r$ if $r = \dim \mathcal{H}_A$, and $\{\lambda_i^2\}_{i=1}^r \cup \{0\}$ if $r < \dim \mathcal{H}_A$.

Similarly, the spectrum of $\rho_B$ is $\{\lambda_i^2\}_{i=1}^r$ if $\dim \mathcal{H}_B = r$ and $\{\lambda_i^2\}_{i=1}^r \cup \{0\}$ if $r < \dim \mathcal{H}_B$.

Thus the two spectrums coincide, up to the point $\{0\}$. $\square$

Remark: How to check $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is of the form $|\psi\rangle = |\phi\rangle_A \otimes |\chi\rangle_B$?

(a) Write it in mx form, check the rank $=1$.

(b) $\mathrm{tr}_B |\psi\rangle\langle\psi|$ is rank-1

Thm: Any density matrix can be written as the marginal of a pure state.

Note: This is called purification.

Proof: consider $\rho = \sum_{i=1}^{r} \lambda_i |\psi_i\rangle\langle\psi_i| \in B(\mathcal{H})$

eg. eigen decomp., but any pure ensemble decomp. works. Let $\mathcal{K} = \mathbb{C}^r$, and define

$$|\psi\rangle = \sum_{i=1}^{r} \sqrt{\lambda_i} |\psi_i\rangle_{\mathcal{H}} \otimes |i\rangle_{\mathcal{K}}$$

Then

$$\begin{aligned} \mathrm{tr}_K |\psi\rangle\langle\psi| &= \mathrm{tr}_K\left(\sum_{ij} \sqrt{\lambda_i\lambda_j}\, |\psi_i\rangle\langle\psi_j| \otimes |i\rangle\langle j|\right) \\ &= \sum_{ij} \sqrt{\lambda_i\lambda_j}\, |\psi_i\rangle\langle\psi_j| \underbrace{\mathrm{tr}(|i\rangle\langle j|)}_{\delta_{ij}} \\ &= \sum_i \lambda_i |\psi_i\rangle\langle\psi_i| = \rho. \end{aligned}$$

<u>Note</u>: we can start from diff. ensemble decomp. $\Rightarrow$ we arrive at diff. purifications. (e.g. $n$ is not necessarily the rank of $\rho$, so even $\dim \mathcal{K}$ might be different.)

<u>Thm</u>: Let $|\psi\rangle \in \mathcal{H}\otimes\mathcal{K}_1$, $|\phi\rangle \in \mathcal{H}\otimes\mathcal{K}_2$. These states are the purifications of the same state,

$$\mathrm{tr}_{\mathcal{K}_1} |\psi\rangle\langle\psi| = \mathrm{tr}_{\mathcal{K}_2} |\phi\rangle\langle\phi|$$

iff there is a map $\mathcal{U}: \mathcal{K}_1 \to \mathcal{K}_2$ s.t.

$|\phi\rangle = (\mathbb{1}\otimes\mathcal{U})|\psi\rangle$, and

$\mathcal{U}\mathcal{U}^\dagger$ and $\mathcal{U}^\dagger\mathcal{U}$ are projectors.

<u>Proof</u>: If $|\psi\rangle = \sum_{i=1}^{r} \lambda_i |\psi_i^{(1)}\rangle \otimes |\psi_i^{(2)}\rangle$ is the Schmidt decomp. of a purification of $\rho$, then

$\rho = \sum_{i=1}^{r} \lambda_i^2 |\psi_i^{(1)}\rangle\langle\psi_i^{(1)}|$ is an ensemble decomp. of $\rho$. Using the relation between two ensemble decompositions, we obtain that if

$|\phi\rangle = \sum_{i=1}^{s} \mu_i^2 |\phi_i^{(1)}\rangle \otimes |\phi_i^{(2)}\rangle$, then

$\exists\ \hat{U} : \mathbb{C}^r \to \mathbb{C}^s$ s.t. $\hat{U}^\dagger \hat{U}$ and $\hat{U}\hat{U}^\dagger$ are projectors and

$$\mu_i |\phi_i^{(1)}\rangle = \sum_j \hat{U}_{ij}\, \lambda_j |\psi_j^{(1)}\rangle .$$

Let $U = \sum_{i=1}^{s} \sum_{j=1}^{r} \hat{U}_{ij} |\phi_i^{(2)}\rangle\langle \psi_j^{(2)}|$ .

Then

$$UU^\dagger = \sum_{ijkl} \hat{U}_{ij}\, \overline{\hat{U}}_{kl}\, |\phi_i^{(2)}\rangle\langle\psi_j^{(2)}|\psi_l^{(2)}\rangle\langle\psi_k^{(2)}|$$

$$= \sum_{ikj} \underbrace{\hat{U}_{ij}(\hat{U}^\dagger)_{jk}}_{\text{projector}} |\phi_i^{(2)}\rangle\langle\psi_k^{(2)}|$$

$$= \sum_{ik} P_{ik} |\phi_i^{(2)}\rangle\langle\psi_k^{(2)}| .$$

As $P$ is a projector and $\{|\phi_i^{(2)}\rangle\}$ are orthonormal and $\{|\psi_k^{(2)}\rangle\}$ are orthonormal, $UU^\dagger$ is a projector. Same way $U^\dagger U$ is a projector as well.

☆ End

Let us calculate now

$$(\mathbb{1} \otimes U)|\psi\rangle = \sum_{ijk} \mu_i\, |\psi_i^{(1)}\rangle \otimes \hat{U}_{jk}\, |\phi_j^{(2)}\rangle \underbrace{\langle\psi_k^{(2)}|\psi_i^{(2)}\rangle}_{\delta_{ki}}$$

$$= \sum_{ij} \underbrace{\lambda_i\, \hat{U}_{ji}\, |\psi_i^{(1)}\rangle}_{\mu_j |\phi_j^{(1)}\rangle} \otimes |\phi_j^{(2)}\rangle$$

$$= \sum_j \mu_j\, |\phi_j^{(1)}\rangle \otimes |\phi_j^{(2)}\rangle .$$

The other direction: assume $\mathcal{U}\mathcal{U}^\dagger$ and $\mathcal{U}^\dagger\mathcal{U}$ are projectors and that

$$(\mathbb{1}\otimes\mathcal{U})|\psi\rangle = |\phi\rangle .$$

Consider the Schmidt decomp. of $|\phi\rangle$ and $|\psi\rangle$:

$$|\psi\rangle = \sum_{i=1}^{r} \lambda_i |\psi_i^{(1)}\rangle \otimes |\psi_i^{(2)}\rangle$$

$$|\phi\rangle = \sum_{i=1}^{s} \mu_i |\phi_i^{(1)}\rangle \otimes |\phi_i^{(2)}\rangle$$

We can extend $\{|\psi_i^{(2)}\rangle\}$ to be ONB, and $\{|\phi_i^{(2)}\rangle\}$ as well. In that basis we can write

$$\mathcal{U} = \sum_{ij} \hat{u}_{ij} |\phi_i^{(2)}\rangle\langle\psi_i^{(2)}| .$$

Then

$$(\mathbb{1}\otimes\mathcal{U})|\psi\rangle = \sum_{ijk} \lambda_i |\psi_i^{(1)}\rangle \otimes \hat{u}_{jk} |\phi_j^{(2)}\rangle \underbrace{\langle\psi_k^{(2)}|\psi_i^{(1)}\rangle}_{\delta_{ik}} =$$

$$= \sum_{ij} \lambda_i |\psi_i^{(1)}\rangle \otimes \hat{u}_{ji} |\phi_j^{(2)}\rangle \overset{!}{=} |\phi\rangle .$$

Comparing this to the Schmidt decomp. of $|\phi\rangle$ we obtain

$$\sum_i \mu_j |\phi_j^{(1)}\rangle \otimes |\phi_j^{(2)}\rangle = \sum_{ij} \lambda_i \hat{u}_{ji} |\psi_i^{(1)}\rangle \otimes |\phi_j^{(2)}\rangle ,$$

and thus that

$$\mu_j |\phi_j^{(1)}\rangle = \sum_i \lambda_i \hat{u}_{ji} |\psi_i^{(1)}\rangle .$$

As the matrix $\hat{u}$ is s.t. $\hat{u}\hat{u}^\dagger$ and $\hat{u}^\dagger\hat{u}$ are projectors, this implies that

$$\sum_i \lambda_i^2 \, |\psi_i^{(1)}\rangle\langle\psi_i^{(1)}| = \sum_i \mu_i^2 \, |\phi_i^{(1)}\rangle\langle\phi_i^{(1)}|,$$

i.e. the reduced density of $|\psi\rangle\langle\psi|$ and $|\phi\rangle\langle\phi|$ is the same. $\square$

## 9th lecture: 30/10/23

<u>Recap</u>: Composite systems: $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$,

$\rho_{AB} \in \mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_B)$.

We have defined the partial trace operation:

If $\rho_{AB} = \sum_i X_i \otimes Y_i$, then

$$\rho_A = \mathrm{tr}_B \, \rho_{AB} = (\mathrm{id} \otimes \mathrm{tr})(\rho_{AB}) = \sum_i \underset{\substack{\uparrow \\ \text{matrix}}}{X_i} \, \underset{\substack{\uparrow \\ \in \mathbb{C}}}{\mathrm{tr}(Y_i)}$$

$$\rho_B = \mathrm{tr}_A \, \rho_{AB} = (\mathrm{tr} \otimes \mathrm{id})(\rho_{AB}) = \sum_i \underset{\substack{\uparrow \\ \in \mathbb{C}}}{\mathrm{tr}(X_i)} \cdot \underset{\substack{\uparrow \\ \text{matrix}}}{Y_i}$$

$\rho_A, \rho_B$ are called the <u>reduced densities</u> of $\rho_{AB}$.

Remark: $\rho_{AB} \neq \rho_A \otimes \rho_B$ in general.

We have also seen that every $\rho \in \mathcal{B}(\mathcal{H})$ arises as the reduced density of some <u>pure</u> state

$|\psi\rangle \in \mathcal{H} \otimes \mathcal{K}$, $\| |\psi\rangle \| = 1$.

$|\psi\rangle$ is called a <u>purification</u> of $\rho$ if $\mathrm{tr}_{\mathcal{K}} |\psi\rangle\langle\psi| = \rho$.

We were proving the

**Thm**: Let $|\psi\rangle \in \mathcal{H} \otimes \mathcal{K}_1$ and $|\phi\rangle \in \mathcal{H} \otimes \mathcal{K}_2$ be states. (that is, $\| |\psi\rangle \| = \| |\phi\rangle \| = 1$.) Then the following are equivalent:

(1) $\mathrm{tr}_{\mathcal{K}_1} |\psi\rangle\langle\psi| = \mathrm{tr}_{\mathcal{K}_2} |\phi\rangle\langle\phi|$

(2) $\exists \mathcal{U} \in \mathrm{Lin}(\mathcal{K}_2, \mathcal{K}_1)$ such that

(a) $\mathcal{U}\mathcal{U}^\dagger$ and $\mathcal{U}^\dagger\mathcal{U}$ are projectors

(b) $|\psi\rangle = (\mathrm{id} \otimes \mathcal{U}) |\phi\rangle$.

**Proof**: $1 \Rightarrow 2$: We have seen that if

$$\left.\begin{aligned} |\psi\rangle &= \sum_{i=1}^{r} \lambda_i |\psi_i^{(1)}\rangle \otimes |\psi_i^{(2)}\rangle \\ |\phi\rangle &= \sum_{i=1}^{s} \mu_i |\phi_i^{(1)}\rangle \otimes |\phi_i^{(2)}\rangle \end{aligned}\right\} \text{Schmidt decomp.}$$

Then

$$(1) \Rightarrow \sum_i \lambda_i^2 |\psi_i^{(1)}\rangle\langle\psi_i^{(1)}| = \sum_i \mu_i^2 |\phi_i^{(1)}\rangle\langle\phi_i^{(1)}|$$

And thus $\exists\, \hat{\mathcal{U}} \in \mathrm{Lin}(\mathbb{C}^s, \mathbb{C}^r)$ s.t.

$$\lambda_i |\psi_i^{(1)}\rangle = \sum_j \hat{\mathcal{U}}_{ij} \mu_j |\phi_j^{(1)}\rangle$$ and $\hat{\mathcal{U}}^\dagger\hat{\mathcal{U}}$, $\hat{\mathcal{U}}\hat{\mathcal{U}}^\dagger$ are proj.

We can define then

$$\mathcal{U} = \sum_{ij} \hat{\mathcal{U}}_{ij} |\psi_i^{(2)}\rangle\langle\phi_j^{(2)}|$$

We have seen that $\mathcal{U}\mathcal{U}^\dagger$ and $\mathcal{U}^\dagger\mathcal{U}$ are projectors (as $\hat{\mathcal{U}}$ is a projector and

$\hat{\mathcal{U}}$ is simply the mx describing $\mathcal{U}$ in the ONB $|\phi_i^{(2)}\rangle, |\psi_i^{(2)}\rangle$.)

Let us show that

$$(\mathbb{1}\otimes\mathcal{U})|\phi\rangle = \sum_i \mu_i |\phi_i^{(1)}\rangle \otimes \mathcal{U}|\phi_i^{(2)}\rangle$$

$$= \sum_{ijk} \mu_i |\phi_i^{(1)}\rangle \otimes \hat{\mathcal{U}}_{jk} |\psi_j^{(2)}\rangle\langle\phi_k^{(2)}|\phi_i^{(2)}\rangle$$

$$= \sum_{ij} \mu_i \hat{\mathcal{U}}_{ji} |\phi_i^{(1)}\rangle \otimes |\psi_j^{(2)}\rangle$$

$$= \sum_j \lambda_j |\psi_j^{(2)}\rangle \otimes |\phi_j^{(2)}\rangle.$$

$\boxed{2 \Rightarrow 1}$ Similar: if

$|\psi\rangle = (\mathbb{1}\otimes\mathcal{U})|\phi\rangle$, then

$$\sum_i \lambda_i |\psi_i^{(1)}\rangle \otimes |\psi_i^{(2)}\rangle = \sum_j \mu_j |\phi_j^{(1)}\rangle \otimes \mathcal{U}|\phi_j^{(2)}\rangle$$

And thus $\mu_j \mathcal{U}|\phi_j^{(2)}\rangle = \sum_k \lambda_k \hat{\mathcal{U}}_{kj} |\psi_k^{(2)}\rangle$.

As $\mathcal{U}^\dagger\mathcal{U}$ and $\mathcal{U}\mathcal{U}^\dagger$ is a projector and $\hat{\mathcal{U}}$ is the mx describing $\mathcal{U}$, $\hat{\mathcal{U}}^\dagger\hat{\mathcal{U}}$ and $\hat{\mathcal{U}}\hat{\mathcal{U}}^\dagger$ is a projector as well.

We can then write:

$$\sum_i \lambda_i |\psi_i^{(1)}\rangle \otimes |\psi_i^{(2)}\rangle = \sum_{jk} \mu_j |\phi_j^{(1)}\rangle \otimes \hat{u}_{kj} |\psi_k^{(2)}\rangle$$

And thus $\forall i$

$$\lambda_i |\psi_i^{(1)}\rangle = \sum_j \mu_j \hat{u}_{ij} |\phi_j^{(1)}\rangle$$

Therefore

$$\sum_i \lambda_i^2 |\psi_i^{(1)}\rangle\langle\psi_i^{(1)}| = \sum_j \mu_j^2 |\phi_j^{(1)}\rangle\langle\phi_j^{(1)}| .$$

$\square$

We have thus understood the redundancy of the Schmidt decomposition.

<u>Measurements on a composite system</u>

Let $\rho_{AB} \in \mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_B)$ be the density mx of a composite system.

Let $\{\Pi_i\}_{i=1}^n \subseteq \mathcal{B}(\mathcal{H}_B)$ be a measurement on the B system.

We say that B does the above measurement on the composite system for the measurement described by the operators $\{\mathbb{1}\otimes M_i\}_{i=1}^{n}$.

<u>Note:</u> $\sum_i M_i^\dagger M_i = \mathbb{1}_B \Rightarrow \sum_i (\mathbb{1}\otimes M_i^\dagger)(\mathbb{1}\otimes M_i) = \mathbb{1}_A\otimes\mathbb{1}_B$
$= \mathbb{1}_{AB}$.

That is, the outcome probabilities and post meas. states after B measuring $\{M_i\}_{i=1}^{n}$ is

$$p_i = \mathrm{tr}_{AB}\left((\mathbb{1}\otimes M_i)\rho_{AB}(\mathbb{1}\otimes M_i^\dagger)\right) \overset{!}{=} \mathrm{tr}(M_i \rho_B M_i^\dagger)$$

$$\rho_i = \frac{1}{p_i}(\mathbb{1}\otimes M_i)\rho_{AB}(\mathbb{1}\otimes M_i)$$

<u>Example:</u>

- $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
- B measures $Z$.

"Measuring $Z$": measuring in the eig. basis of $Z$: measurement op: $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$.
B measures $Z$: meas. op. on the composite system is $\mathbb{1}\otimes|0\rangle\langle 0|$ and $\mathbb{1}\otimes|1\rangle\langle 1|$.

Outcome: +1/−1 or 0/1.

$$P_0 = \mathrm{tr}\Big( (\mathbb{1}\otimes|0\rangle\langle 0|)\, |\psi\rangle\langle\psi|\, (\mathbb{1}\otimes|0\rangle\langle 0|) \Big)$$

$$= \langle 0|\rho_B|0\rangle = \frac{1}{2}$$

$$P_1 = \langle 1|\rho_B|1\rangle = \frac{1}{2}.$$

Post meas. state:

$$\frac{1}{p}(\mathbb{1}\otimes|0\rangle\langle 0|)\,|\psi\rangle\langle\psi|\,(\mathbb{1}\otimes|0\rangle\langle 0|)$$

How to do this calculation cleverly?

(1) It is a pure state, so we only need

$$(\mathbb{1}\otimes|0\rangle\langle 0|)\,|\psi\rangle$$

(2) Now it's easier:

$$|\psi\rangle = \frac{1}{2}(|00\rangle + |11\rangle)$$

So

$$(\mathbb{1}\otimes|0\rangle\langle 0|)|\psi\rangle = \frac{1}{2}\Big((\mathbb{1}\otimes|0\rangle\langle 0|)|00\rangle + (\mathbb{1}\otimes|0\rangle\langle 0|)|11\rangle\Big)$$

$$= \frac{1}{2}\big(|0\rangle\otimes|0\rangle\underbrace{\langle 0|0\rangle} + |1\rangle\otimes|1\rangle\underbrace{\langle 0|1\rangle}_{0}\big)$$

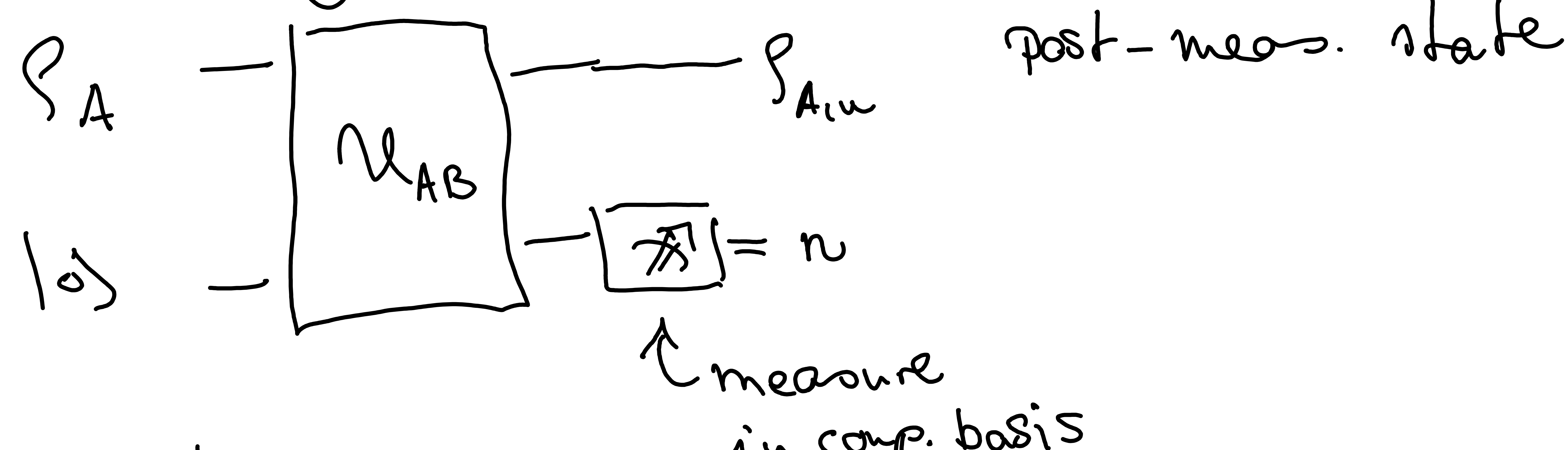
$$= \frac{1}{2}|00\rangle. \Rightarrow \psi_0 = |00\rangle$$

<u>HW</u>: think through what $\psi_1$ is.

Note: $(\mathbb{1}\otimes|0\rangle\langle 0|)|\psi\rangle = |v\rangle\otimes|0\rangle$ for some vector $v$. $\boxed{\text{HW ?}}$

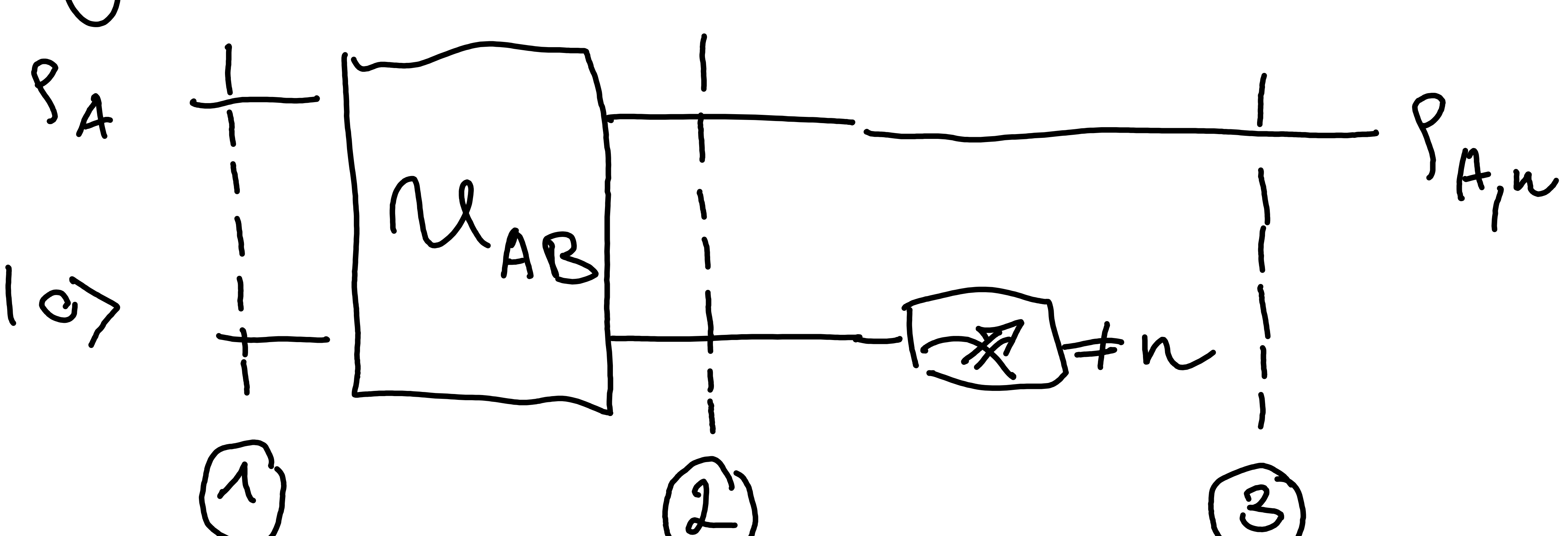
People usualy only introduce proj. measurements.
Let us show a situation where proj. measurement is not enough / the origin of POVMS. Measure comp. system after time evol.

Graphically:




When describing only System A, the description is a POVM.

<u>Analysis</u>:




1. The state is $\rho_A \otimes |0\rangle\langle 0|$
2. Afther the unitary, the state becomes
$$U_{AB} \left( \rho_A \otimes |0\rangle\langle 0| \right) U_{AB}^{\dagger} .$$
3. Measurement in the comp. basis in the System B:

The meas. ops. are $\{\mathbb{1}\otimes|i\rangle\langle i|\}_{i=1}^{n}$, so

$$p_i = \mathrm{tr}\{(\mathbb{1}\otimes|i\rangle\langle i|)\, U_{AB}\, (\rho_A\otimes|0\rangle\langle 0|)\, U_{AB}^{+}\, (\mathbb{1}\otimes|i\rangle\langle i|)\}.$$

To simplify, write $U_{AB} = \sum_i X_i \otimes Y_i$.

We then obtain:

$$p_i = \sum_{jk} \mathrm{tr}\{ X_j \rho_A X_k^{+} \otimes |i\rangle\langle i| Y_j |0\rangle\langle 0| Y_k^{+} |i\rangle\langle i|\}$$

$$= \sum_{jk} \mathrm{tr}\{ X_j \rho_A X_k^{+}\} \cdot \mathrm{tr}\{|i\rangle\langle i|\} \cdot \langle i|Y_j|0\rangle \cdot \langle 0|Y_k^{+}|i\rangle$$

<u>Note:</u> $\mathrm{tr}_{AB} = \mathrm{tr}_A \otimes \mathrm{tr}_B$ as $\mathrm{tr}_{AB}(\rho) = \sum_{ij} \langle ij|\rho|ij\rangle$,

or if $\rho = \sum_k \rho_k^{(1)} \otimes \rho_k^{(2)}$, then

$$\mathrm{tr}\,\rho = \sum_k \sum_i \langle i|\rho_k^{(1)}|i\rangle \cdot \sum_j \langle j|\rho_k^{(2)}|j\rangle$$

$$= \sum_k \mathrm{tr}\,\rho_k^{(1)} \cdot \mathrm{tr}\,\rho_k^{(2)}.$$

Let now

$$\Pi_i = \sum_j X_j \langle i|Y_j|0\rangle.$$

Then

$$p_i = \mathrm{tr}\{\Pi_i \rho \Pi_i^{+}\}.$$

Similarly,

$$\rho_{A,i} = \mathrm{tr}_B\{\mathbb{1}\otimes|i\rangle\langle i|\, U_{AB}\, (\rho_A\otimes|0\rangle\langle 0|)\, U_{AB}^{+}\, (\mathbb{1}\otimes|i\rangle\langle i|)\}$$

$$= \sum_{jk} X_j \rho_A X_k^{+} \langle i|Y_j|0\rangle\langle 0|Y_k^{+}|i\rangle$$

$$= \Pi_i \rho_A \Pi_i^{+}.$$

Sanity check: $\sum_i M_i^\dagger M_i = \mathbb{1}$ :

$$\sum_i M_i^\dagger M_i = \sum_{ijk} X_k^\dagger X_j \langle 0| Y_k^\dagger |i\rangle\langle i| Y_j |0\rangle = \mathbb{1}.$$

Actually, any POVM can arise this way:

Consider

$$X = \sum_{i=1}^{n} M_i \otimes |i\rangle\langle 0| = \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{pmatrix}$$

Then

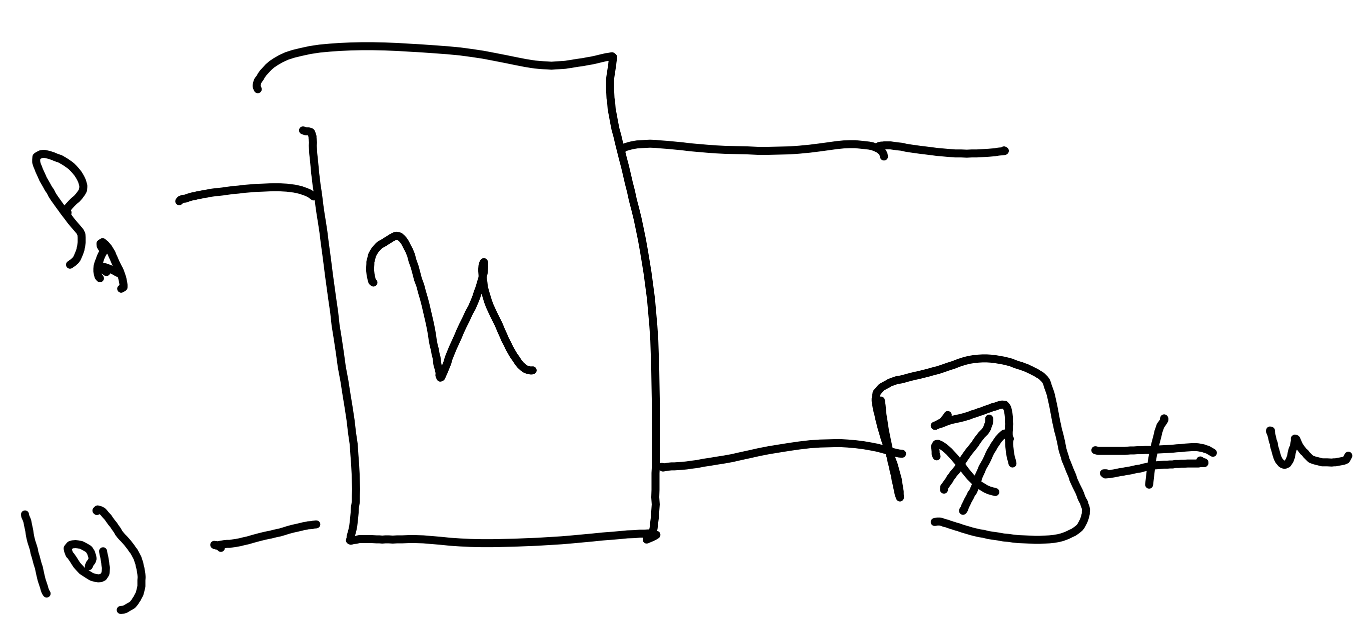
$$X^\dagger X = \sum_{ij} M_i^\dagger M_j \otimes |0\rangle\langle i|j\rangle\langle 0|$$

$$= \mathbb{1} \otimes |0\rangle\langle 0|$$

Note that this means that the columns are orthogonal. We can thus complete it to a unitary:

$$\mathcal{U} = \begin{pmatrix} M_1 & - & \cdots & - \\ M_2 & - & & - \\ \vdots & \vdots & & \\ M_n & - & & - \end{pmatrix}$$

Then




results in the POVM.

# 10th lecture: 3/11/2023

**Reminder:**

→ Composite systems are described on a tensor product Hilbert space.

→ Partial trace if $\rho \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$, then

$$\rho_A = \mathrm{tr}_B\, \rho = (\mathrm{id} \otimes \mathrm{tr})(\rho)$$

$$\rho_B = \mathrm{tr}_A\, \rho = (\mathrm{tr} \otimes \mathrm{id})(\rho).$$

$\rho_A, \rho_B$ are called the reduced densities or marginals of $\rho$.

Concrete calculation: if

$$\rho = \sum_{ij} |i\rangle\langle j| \otimes \rho_{ij}^R$$

$$\rho_B = \sum_i \rho_{ii}^R.$$

In mx notation:

$$\rho = \begin{pmatrix} \rho_{00}^R & \rho_{01}^R & \cdots & \\ \rho_{10}^R & & \ddots & \\ \vdots & & \vdots & \\ \cdots & & & \rho_{nn}^R \end{pmatrix}$$

where $\rho_{ij}^L$ are $\dim(\mathcal{H}_A) \times \dim(\mathcal{H}_A)$ matrices.

Then

$\rho_B = \sum_i \rho^L_{ii}$ is the sum of the matrices that are in the diagonal.

Similarly,

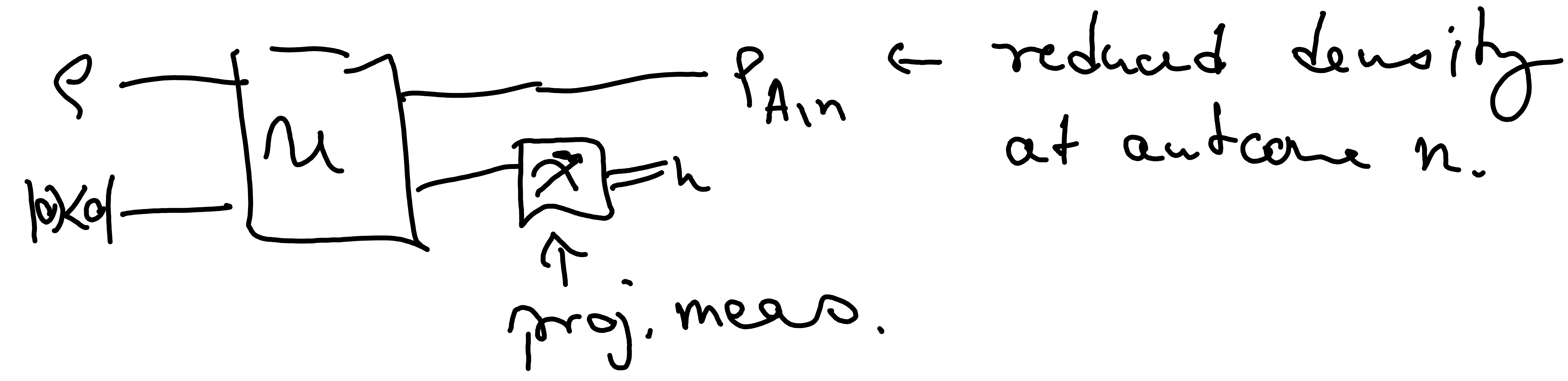
$$\rho_A = \sum_{ij} |j\rangle\langle j| \cdot \mathrm{tr}(\rho^R_{ij}),$$

So $\rho_A$ is the mx that one obtains when tracing each block.

→ We have seen that ∀ density mx is reduced density of some pure state

→ Also that given a density mx, it has several purifications, but they are related to eachother.

→ We have seen that ∀ POVM arises as



## Today: time evolution, CP maps:

We have learned:

→ Closed system: $\rho \mapsto U \rho U^{\dagger}$.

TP
Positive

→ Open system: $\rho \mapsto T(\rho)$, where

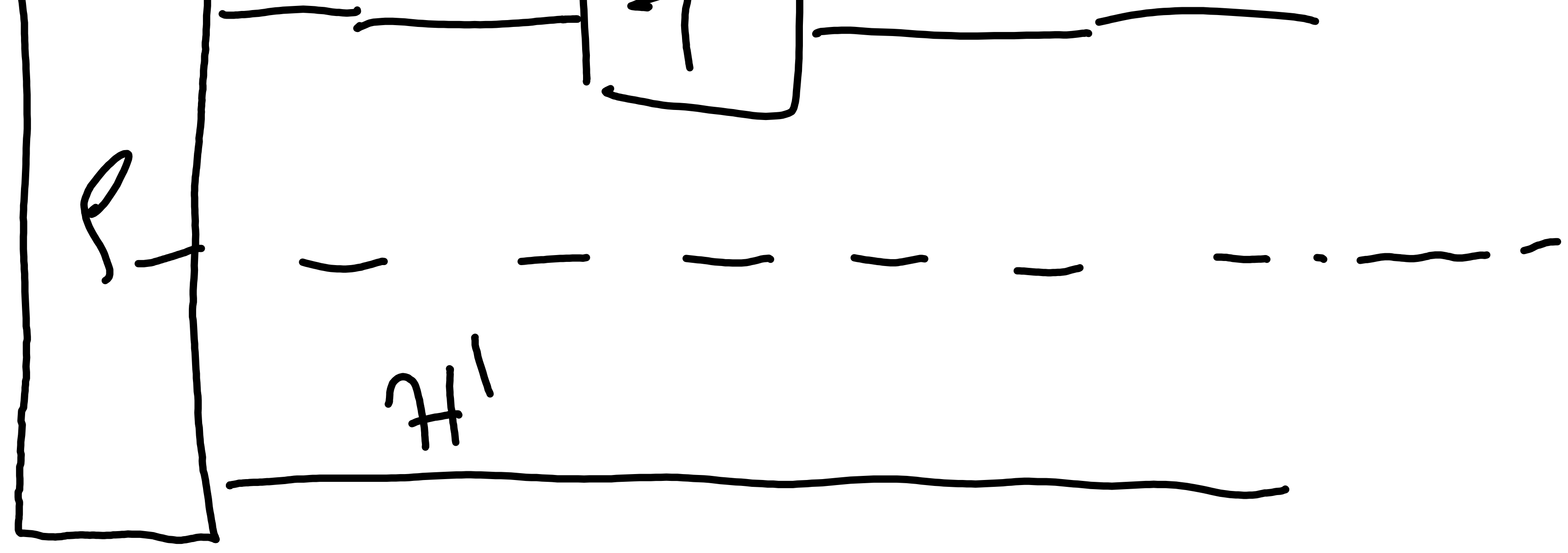
T is a CPTP map.

Def: CPTP map : $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$

- linear
- $\operatorname{tr} T(X) = \operatorname{tr}(X) \quad \forall X \in \mathcal{B}(\mathcal{H})$
- For all H-space $\mathcal{H}'$ the map

$$(T \otimes \mathrm{id})(\rho) \geq 0 \quad \text{if} \quad \rho \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}') \geq 0.$$

First of all, why the requirement $(T \otimes \mathrm{id})(\rho) \geq 0$?




} this s/b also positive as tensoring w/ id is allowed.

Why is $T(\rho) \geq 0$ if $\rho \geq 0$ not enough?

We say that $T(\rho) \geq 0$ if $\rho \geq 0$ is a positive map.

Thm: The transposition map,

$$T(|i\rangle\langle j|) = |j\rangle\langle i|$$

is positive, but not completely positive.

Proof: Let us prove first that $T$ is positive.

Let $\rho \geq 0$. Then we can write $\rho = XX^+$. Its transpose is

$$T(\rho) = (X^+)^T X^T = (X^T)^+ X^T,$$ so

$(X^+)^T = (X^T)^+ = \bar{X}$, the complex conjugate of $X$.

So

$$T(\rho) = Y^+ Y$$ with $Y = X^T$, thus $T(\rho) \geq 0$.

Let us show that $T$ is not CP.

For that, consider the state

$\rho = |\Omega\rangle\langle\Omega|$ with $|\Omega\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)$.

That is,

$$\rho = \frac{1}{2}\big(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\big).$$

$$= \frac{1}{2}\big(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 0| \otimes |1\rangle\langle 0| +$$
$$+ |1\rangle\langle 1| \otimes |1\rangle\langle 1|\big).$$

Then

$$(id \otimes T)(\rho) = \frac{1}{2}\big(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |1\rangle\langle 0|$$
$$+ |1\rangle\langle 0| \otimes |0\rangle\langle 1| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|\big)$$

$$= \frac{1}{2}\big(|00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 11|\big)$$

This is not positive, as

$$\rho|01\rangle = |10\rangle \quad \text{and} \quad \rho|10\rangle = |01\rangle$$

and thus

$$\rho\big(|01\rangle - |10\rangle\big) = (-1)\cdot\big(|01\rangle - |10\rangle\big),$$

i.e. $(-1)$ is an eigenvalue. $\square$

<u>Note:</u> in matrix notation, these objects are:

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$(id \otimes T)(\rho) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ not positive.}$$

Do we have CP (completely positive) maps at all? What about unitary?

<u>Thm</u>: The unitary evolution map,

$$T(\rho) = U \rho U^\dagger \text{ is CP.}$$

<u>Proof</u>: Let us consider $\rho \in B(\mathcal{H}) \otimes B(\mathcal{K}')$.

Then

$$(T \otimes id)(\rho) = (U \otimes \mathbb{1}) \rho (U^\dagger \otimes \mathbb{1}).$$

Assume $\rho$ is positive, $\rho = XX^\dagger$.

Then

$$(U \otimes 1) X X^\dagger (U^\dagger \otimes 1) = YY^\dagger \text{ w/ } Y = (U \otimes 1) X.$$

What other example can we have?

<u>Example</u>:

- Take unitaries $U_i$, probabilities $p_i$.
- With proba $p_i$ do evolution $U_i$:

$$T(\rho) = \sum_i p_i U_i \rho U_i^\dagger.$$

This is also CP, as

$$(T \otimes id)(\rho_{AB}) = \sum_i \sqrt{p_i} (U_i \otimes \mathbb{1}) \rho (U_i^\dagger \otimes \mathbb{1}) \cdot \sqrt{p_i},$$

and thus if $\rho_{AB} = XX^+$, then

$$(T \otimes id)(\rho) = \sum_i Y_i Y_i^+ \geq 0, \text{ where}$$

$$Y_i = \sqrt{p_i}\, U_i .$$

This can be generalized:

<u>Def</u>: If the map $T : B(\mathcal{H}) \to B(\mathcal{K})$ is of the form

$$T(\rho) = \sum_i A_i \rho A_i^+ \quad (*)$$

w/ $A_i : \mathcal{H} \to \mathcal{K}$

then $(*)$ is called the Kraus representation of $T$.

<u>Thm</u>: If a map $B(\mathcal{H}) \to B(\mathcal{K})$ admits a Kraus representation, then it is CP.

<u>Proof</u>: $T(\rho) = \sum_i A_i \rho A_i^+$.

$$(T \otimes id)(\rho_{AB}) = \sum_i (A_i \otimes \mathbb{1})\, \rho_{AB} (A_i^+ \otimes \mathbb{1})$$

again, if $\rho_{AB} = XX^+$, then

$$(T \otimes id)(\rho_{AB}) = YY^+ \text{ with}$$

$$Y = \left( (A_1 \otimes \mathbb{1}) X, (A_2 \otimes \mathbb{1}) X, \ldots, (A_n \otimes \mathbb{1}) X \right).$$

$\square$

<u>Note</u>: Example CP maps :

- trace : $\rho \mapsto \mathrm{tr}(\rho)$
- partial trace $\rho_{AB} \mapsto \mathrm{tr}_B(\rho)$
- tensoring w/ positive op. $\rho \mapsto \rho \otimes \sigma$.

<u>Lemma</u>: If $T, \mathcal{E}$ admit a Kraus representation, then $T \circ \mathcal{E}$ as well.

<u>Proof</u>: if
$$T(\rho) = \sum_i A_i \rho A_i^+$$
$$\mathcal{E}(\rho) = \sum_i B_i \rho B_i^+$$

Then
$$T \circ \mathcal{E}(\rho) = \sum_{ij} A_i B_j \rho B_j^+ A_i^+ ,$$
and this is a Kraus repr. w/ operators $\{A_i B_j\}_{ij}$.

$\square$

<u>Lemma</u>. $T$ w/ Kraus repr. $\{A_i\}_{i=1}^n$ is trace preserving (TP) iff
$$\sum_i A_i^+ A_i = \mathbb{1}.$$

<u>Proof</u>: $\mathrm{tr}(T(\rho)) = \mathrm{tr}\left(\sum_i A_i \rho A_i^+\right) = \mathrm{tr}\left(\rho \sum_i A_i^+ A_i\right)$.
it is $1$ for all $\rho$ w/ $\mathrm{tr}\,\rho = 1$ iff $\sum_i A_i^+ A_i = \mathbb{1}$. $\square$

Let us show now that every CP map admits a Kraus representation,

<u>Lemma</u>: Let $|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_i |ii\rangle \in \mathcal{H} \otimes \mathcal{H}$.
Then for any $O \in \mathcal{B}(\mathcal{H})$,

$$(\mathbb{1} \otimes O)|\Omega\rangle = (O^T \otimes \mathrm{id})|\Omega\rangle.$$

<u>Proof</u>: Let us write

$$O = \sum_{ij} O_{ij} |i\rangle\langle j| .$$

Then

$$\begin{aligned}
(\mathbb{1} \otimes O)|\Omega\rangle &= \frac{1}{\sqrt{d}} \sum_{ijk} O_{ij} \cdot |k\rangle \otimes |i\rangle\langle j|k\rangle \\
&= \frac{1}{\sqrt{d}} \sum_{ik} O_{ik} |k\rangle \otimes |i\rangle . \\
&= \frac{1}{\sqrt{d}} \sum_{ijk} \underbrace{O_{jk} |k\rangle\langle j|}_{O^T} i\rangle \otimes |i\rangle \\
&= (O^T \otimes \mathbb{1})|\Omega\rangle .
\end{aligned}$$

☆ End. $\square$

11th lecture 6/11/2023

Reminder:

- Positive matrices / operators

$M \in B(\mathcal{H}) \geq 0$ iff $\langle\varphi|M|\varphi\rangle \geq 0 \ \forall \varphi \in \mathcal{H}$.

- Sometimes we consider linear maps that map matrices to matrices, e.g. $\rho \mapsto U\rho U^\dagger$.

Such a map is called positive / positivity preserving if it maps positive objects to positive: $T: B(\mathcal{H}) \to B(\mathcal{K})$ is positive if

$$T(X) \geq 0 \quad \forall X \in B(\mathcal{H}), X \geq 0.$$

- From time evolution and any operation (!) we require not only positivity, but also complete positivity:

$$(T \otimes id)(\rho) \geq 0 \text{ for all } \rho \in B(\mathcal{H} \otimes \mathcal{H}'), \rho \geq 0.$$

This is because we want:

- starting from $\rho \in B(\mathcal{H}) \otimes B(\mathcal{H}')$
- time-evolving on $\mathcal{H}$
- but doing nothing on $\mathcal{H}'$

should result in a valid state.

We have seen that positive $\subsetneq$ CP :

- CP $\Rightarrow$ positive w/ $\mathcal{H}' \subseteq \mathbb{C}$
- transposition is positive but not CP:

$$(\mathrm{id} \otimes T)(|\Omega\rangle\langle\Omega|) \not\geq 0 \quad \text{in 2D}$$

where $\quad |\Omega\rangle = \frac{1}{\sqrt{d}} \sum_i |ii\rangle.$

We have seen example CP maps:

$$T(\rho) = \sum_i A_i \rho A_i^{+} \quad \text{is CP.}$$

This is what we call Kraus representation.

We want to prove: $\forall$ CP maps are of this form.

<u>Lemma</u>:

$$(O \otimes \mathbb{1})|\Omega\rangle = (\mathbb{1} \otimes O^T)|\Omega\rangle .$$

($|\Omega\rangle$ is the state def. above)

**Lemma**: Let $\dim \mathcal{H} = d$, and

$$|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\, i\rangle \in \mathcal{H} \otimes \mathcal{H}.$$

Let $\mathcal{K}$ be another Hilbert space, $|\phi\rangle \in \mathcal{K} \otimes \mathcal{H}$ arbitrary. Then there is $X \in \mathrm{Lin}(\mathcal{H}, \mathcal{K})$ s.t.

$$|\phi\rangle = (X \otimes \mathrm{id})\, |\Omega\rangle.$$

**Proof**: Write $|\phi\rangle = \sum_{ij} \phi_{ij} |i j\rangle$.

Let $X = \sqrt{d} \sum_{ij} \phi_{ij} |i\rangle\langle j|$,

then

$$(X \otimes \mathrm{id})\, |\Omega\rangle = \sum_{ijk} \phi_{ij} |i\rangle\langle j|k\rangle \otimes |k\rangle$$

$$= \sum_{ij} \phi_{ij} |i\rangle |j\rangle.$$

$\square$

Thm: Every CP map admits a Kraus representation.

Proof: Let $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be CP and assume that $\dim(\mathcal{H}) = d$. Then let $\Omega \in \mathcal{H} \otimes \mathcal{H}$ be defined as
$$|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle .$$
Notice that $\mathrm{tr}_B\{|\Omega\rangle\langle\Omega|\} = \frac{1}{d} \mathbb{1}$.

We obtain thus that
$$\begin{aligned}
T(\rho) &= T\left(\rho \cdot \mathrm{tr}_B(|\Omega\rangle\langle\Omega|)\right) \\
&= T\left(\mathrm{tr}_B\left((\rho \otimes \mathbb{1})|\Omega\rangle\langle\Omega|\right)\right) \\
&= T\left(\mathrm{tr}_B\left((\mathbb{1} \otimes \rho^T)|\Omega\rangle\langle\Omega|\right)\right) \\
&= (T \otimes \mathrm{tr})\left((\mathbb{1} \otimes \rho^T)|\Omega\rangle\langle\Omega|\right) \\
&= (\mathrm{id} \otimes \mathrm{tr}) \circ (T \otimes \mathrm{id})\left((\mathbb{1} \otimes \rho^T)|\Omega\rangle\langle\Omega|\right) \\
&= \mathrm{tr}_B\left\{(\mathbb{1} \otimes \rho^T)(T \otimes \mathrm{id})(|\Omega\rangle\langle\Omega|)\right\}.
\end{aligned}$$

Note now that as $T$ is CP,
$$\underbrace{(T \otimes \mathrm{id})(\underbrace{|\Omega\rangle\langle\Omega|}_{\in\, \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})})}_{\mathcal{B}(\mathcal{K}) \otimes \mathcal{B}(\mathcal{H})} \geq 0.$$

Write

$$(T\otimes \mathrm{id})(|\Omega\rangle\langle\Omega|) = \sum_k |\underbrace{\psi_k}_{\mathcal{K}\otimes\mathcal{H}}\rangle\langle\psi_k|$$

$$= \sum_k (A_k\otimes \mathbb{1})|\Omega\rangle\langle\Omega|(A_k^\dagger\otimes\mathbb{1})$$

Then

$$T(\rho) = \mathrm{tr}_B\{(1\otimes\rho^T)\sum_k (A_k\otimes\mathbb{1})|\Omega\rangle\langle\Omega|(A_k^\dagger\otimes\mathbb{1})\}$$

$$= \sum_k \mathrm{tr}_B\{A_k\otimes\rho^T|\Omega\rangle\langle\Omega|(A_k^\dagger\otimes 1)\}$$

$$= \sum_k \mathrm{tr}_B\{(A_k\rho\otimes\mathbb{1})|\Omega\rangle\langle\Omega|A_k^\dagger\otimes\mathbb{1}\}$$

$$= \sum_k A_k\rho\, \mathrm{tr}_B|\Omega\rangle\langle\Omega|A_k^\dagger$$

$$= \sum_k A_k\rho A_k^\dagger .$$

$\square$

<u>Scholium</u>: $T: B(\mathcal{H})\to B(\mathcal{K})$ lin. map is CP iff

$$(T\otimes \mathrm{id})(|\Omega\rangle\langle\Omega|)\geq 0.$$

<u>Proof</u>: $|\Omega\rangle\langle\Omega|\geq 0$ thus if $T$ is CP,
$(T\otimes\mathrm{id})(|\Omega\rangle\langle\Omega|)\geq 0.$

In the previous proof we have seen that if $(T\otimes id)(|\Omega\rangle\langle\Omega|) \geq 0$, then $T$ admits a Kraus representation. $\square$

<u>Remark</u>: to check that the transposition is not CP, we checked exactly this matrix.

<u>Corollary</u>: (Choi-Jamiolkowsky isomorphism):

The set

$$\{T : B(\mathcal{H}) \to B(\mathcal{K}) \mid T \text{ is CP}\}$$

and the set of positive operators in $B(\mathcal{K})\otimes B(\mathcal{H})$ are in 1-to-1 correspondence

<u>Proof</u>:
- $T \mapsto (T\otimes id)(|\Omega\rangle\langle\Omega|) \geq 0$ & injective.
- Given $X \in B(\mathcal{K})\otimes B(\mathcal{H})$, $X = \sum_{ij} \underbrace{X_{ij}}_{\in B(\mathcal{K})} \otimes \underbrace{|i\rangle\langle j|}_{\in B(\mathcal{H})}$

Define $T(|i\rangle\langle j|) = X_{ij}$. This is CP by constr.

<u>Remark</u>: Time evolution is described by $T : B(\mathcal{H}) \to B(\mathcal{H})$ that is CP and TP.

General operations are described by $B(\mathcal{H}) \to B(\mathcal{K})$ CP and TP. They are also called channels.

**Note**: Channels (i.e. TPCP maps) are in 1-to-1 correspondence w/ density matrices $\rho \in B(\mathcal{K}) \otimes B(\mathcal{H})$ s.t.

$$\mathrm{tr}_{\mathcal{K}}\, \rho = \frac{1}{d}\mathbb{1}.$$

**Proof**: $\mathrm{tr}_{\mathcal{K}}(T\otimes \mathrm{id})(|\Omega\rangle\langle\Omega|) =$

$$= (\mathrm{tr}\circ T\otimes \mathrm{id})(|\Omega\rangle\langle\Omega|)$$

$$= \sum_{ij} \mathrm{tr}(T(x_{ij})) \cdot |i\rangle\langle j|$$

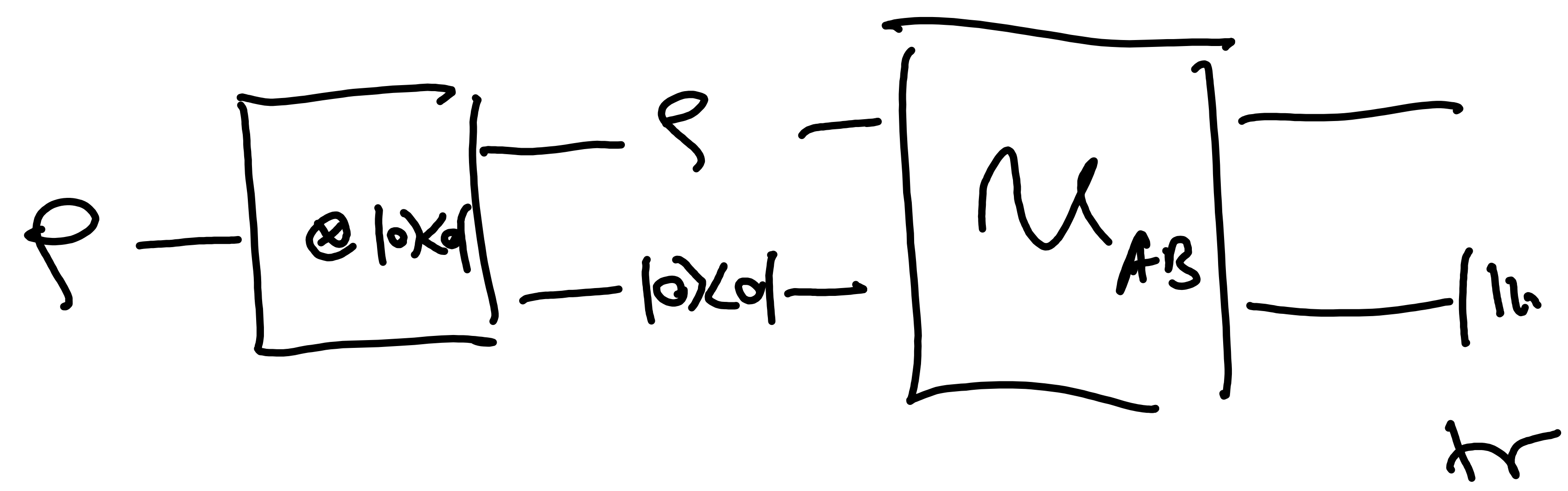
$$\underset{\mathrm{TP}}{=} \sum_{ij} \mathrm{tr}(x_{ij}) \cdot |i\rangle\langle j| = (\mathrm{tr}\otimes \mathrm{id})(|\Omega\rangle\langle\Omega|).$$

$$= \frac{1}{d}\mathbb{1}.$$

Let us show now that TPCP maps arise exactly as

- Adding an ancilla
- Unitary evolution
- Tracing out.

Let us consider now




That is,

$$T(\rho) = \operatorname{tr}_B\Big( U_{AB}(\rho \otimes |0\rangle\langle 0|)\, U_{AB}^{\dagger} \Big).$$

This map is CP, as it admits a Kraus repr. Explicitly, if

$$U_{AB} = \sum_{ij} u_{ij} \otimes |i\rangle\langle j| ,$$

then

$$T(\rho) = \operatorname{tr}_B\Big( \sum_{ijkl} (u_{ij} \otimes |i\rangle\langle j|)(\rho \otimes |0\rangle\langle 0|)(u_{kl}^{\dagger} \otimes |l\rangle\langle k|) \Big)$$

$$= \operatorname{tr}_B\Big( \sum_{ijkl} u_{ij}\, \rho\, u_{kl}^{\dagger} \otimes |i\rangle\langle j|0\rangle\langle 0|l\rangle\langle k| \Big)$$

$$= \sum_i u_{i0}\, \rho\, u_{i0}^{\dagger} .$$

Note that $T$ is also TP, as

$$\sum_i u_{i0}^{\dagger} u_{i0} = (\mathbb{1} \otimes \langle 0|)\, U^{\dagger} U\, (\mathbb{1} \otimes |0\rangle) = \mathbb{1} .$$

The other way: if $\sum_k A_k^\dagger A_k = \mathbb{1}$, then

$$\sum_i |i\rangle \otimes A_i = \begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix}$$

Can be completed to be a unitary

$$\mathcal{U} = \begin{pmatrix} A_0 & \mathcal{U}_{01} & \cdots & \mathcal{U}_{0n} \\ \vdots & \vdots & & \vdots \\ A_n & \mathcal{U}_{n1} & & \mathcal{U}_{nn} \end{pmatrix}.$$

Then

$$\mathrm{tr}_1 \left\{ \mathcal{U} \left( |0\rangle\langle 0| \otimes \rho \right) \mathcal{U}^\dagger \right\} = \sum_k A_k \rho A_k^\dagger.$$

$\square$

So general time evolution is

- adding a system, time evolving, tr out
- CP map: $(T \otimes \mathrm{id})(\rho_{AB}) \geq 0$ if $\rho_{AB} \geq 0$
- Kraus form: $T(\rho) = \sum_k A_k \rho A_k^\dagger$, $\sum_k A_k^\dagger A_k = \mathbb{1}$.

☆ End